A Wavelet Analysis of Random Iterated Function Systems

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Abstract

Random fractal signals obtained as fixed points of Iterated Function Systems (IFS) are good examples of signals which possess discrete scale invariance. In the deterministic case, the problem of approximating a target signal as a fixed point of an IFS with appropriate parameters has been widely studied and is known as the inverse problem [FOR]. So far, the random case has not received much attention and estimating the unknown statictics of the IFS is the focus of our work. To this end, we consider an orthonormal decomposition of the random target signal and extend the results in [MEN] to the random case. Then, we derive the wavelet decompostion of the signal and emphasize the redundancy in the information carried by the wavelet coefficients. This will be used in the last part where we derive an empirical estimator of the variance of the unknown random maps of the IFS in a restrictive case.

1. Definitons

Fractal sets obtained as fixed points of IFS has been widely studied by Barnsley and Hutchinson in [BAR,HUT]. The idea is to apply a contractive operator T (deterministic or random) on an element of a complete metric space. By iterating this procedure, the process converges to a limit point, usually fractal. The existence and uniqueness of the fixed point (also denoted as the attractor of the IFS) is due to the Banach fixed point theorem. We are mainly concerned with random operators T acting on the space $\mathbb{L}_2(\mathbb{X})$ of compactly supported square integrable functions. A more rigorous definition of this space is given in the next section. Generally, one defines the operator T in the following way:

$$(Tf)(x) = \sum_{i=1}^{M} \phi_i[f^i(\omega_i^{-1}(x)), \omega_i^{-1}(x)]\mathbf{1}_{\omega_i(\mathbb{X})}(x) \tag{1.1}$$

where $\mathbb{X}$ is a compact interval of the real line, $\omega_i : \mathbb{X} \to \mathbb{X}$ partition the interval $\mathbb{X}$ into disjoint subintervals and $\phi_i : \mathbb{R} \times \mathbb{X} \to \mathbb{R}$ are random operators Lipschitz in their first variable. Lastly, f^i are independent and identically distributed copies of f. Loosely speaking, this operator realises dilatations, compressions and distorsions of i.i.d. copies of the random function $f \in \mathbb{L}_2(\mathbb{X})$. The limit function obtained is self-similar up to probability distribution.

2. Expansion of Random Functions

Let $\mathbb{X}$ denote a compact support. Without loss of generality, we will consider $\mathbb{X} = [0, 1]$. In the remainder, we will only consider compactly supported signals with finite energy. The space of deterministic square integrable functions is denoted by $L_2(\mathbb{X})$. We endow $L_2(\mathbb{X})$ with its usual metric $l_2(f, g) = (\int_{\mathbb{X}} |f(t) - g(t)|^2 dt)^{\frac{1}{2}}$ where f and g are two elements of $L_2(\mathbb{X})$. To deal with random attractors of IFS and their decomposition over an orthonormal basis, we will need to define new spaces. Let Ω be a sample space. Let us first start with random square integrable signals:

$$\mathbb{L}_2(\mathbb{X}) = \{f(\omega, t), \omega \in \Omega \mid \mathbb{E}_\omega\Big[\int_{\mathbb{X}} |f(\omega, t)|^2 dt\Big] < +\infty\} \tag{2.1}$$

The associated metric is $l_2^*(f, g) = \mathbb{E}_\omega^{\frac{1}{2}}\left[l_2^2(f, g)\right]$. Next we define the space $D_2(\mathbb{N})$ of coefficients of deterministic square integrable functions decomposed on an orthonormal basis. In the remainder of this section, we denote by $\{\varphi_n\}$ an orthonormal basis of $L_2(\mathbb{X})$.:

$$D_2(\mathbb{N}) = \{\mathbf{f} = (f_0, ..., f_j, ...) \mid f(x) = \sum_{k\geq 0} f_k\varphi_k(x), f \in L_2(\mathbb{X})\} \tag{2.2}$$

This space is equipped with the usual metric $d_2(\mathbf{f}, \mathbf{g}) = (\sum_{k\geq 0} |f_k - g_k|^2)^{\frac{1}{2}}$. Likewise, we define the space $\mathbb{D}_2(\mathbb{N})$ which contains the decomposition coefficients of random square integrable functions decomposed on an orthonormal basis. The distance between 2 elements of this space is $d_2^*(\mathbf{f}, \mathbf{g}) = \mathbb{E}_\omega^{\frac{1}{2}}[d_2^2(\mathbf{f}^\omega, \mathbf{g}^\omega)]$. Then for any $f \in \mathbb{L}_2(\mathbb{X})$,

$$f(x) = \sum_{k\geq 0} f_k\varphi_k(x) \tag{2.3}$$

where $f_k =< f, \varphi_k >= \int_{\mathbb{X}} f(x)\varphi_k(x)dx$ are random variables, $\{\varphi_n\}$ remain deterministic. Clearly, the random process f induces a distribution on its coefficients f_k. The purpose of this part is to generalise the work of [FOR] in the random setting. The major result is to show that we can associate with T a contractive operator ϑ acting on the decomposition coefficients of a random function, converging to a fixed point, which is the vector of expansion of the fixed point of T. To this end, let f and g be two elements of $\mathbb{L}_2(\mathbb{X})$ such that $g = Tf$. Consider their decomposition over $\{\varphi_n\}$. $(g(x) = \sum_{k\geq 0} g_k\varphi_k(x))$. Clearly:

$$g_k =< g, \varphi_k >=< Tf, \varphi_k >=< \sum_{i=1}^{M} \phi_i(f^{(i)}(\omega_i^{-1}), \omega_i^{-1})\mathbf{1}_{\omega_i(\mathbb{X})}, \varphi_k >$$

To go further, we need to set a particular expression for ϕ_i. One could think of any possible expression and one of the simplest ones is given by:

$$\phi_i(u, v) = s_i u + \zeta_i(v) \quad s_i < 1 \tag{2.4}$$

where ζ_i is a non-linear function, leading to a nonlinear transformation of the coefficients f_k. We denote by ϑ the mapping which transforms the coefficients of $\mathbf{f}$ into the coefficients of $\mathbf{g}$ after applying T. We have the following result:

Theorem 1. *Let*

- $\{\varphi_n\}$ *be an orthonormal basis of* $L_2(\mathbb{X})$.
- T *be the contractive operator of a random IFS and* f^* *its fixed point.*

Then ϑ *is contractive in the complete metric space* $(\mathbb{D}_2(\mathbb{N}), d_2^*)$ *and for any initial condition* $\mathbf{f}_0$ *(random or non random),* $\vartheta^{\circ k}\mathbf{f}_0 \to \mathbf{f}^*$ *a.s. as* $k \to +\infty$ *where* $\vartheta^{\circ k}$ *is the* k^{th}

iterate of ϑ. Moreover, $\mathbf{f}^*$ is unique in distribution and is the vector of expansion of f^* in the $\{\varphi_n\}$ basis.

The proof is not given here. This theorem shows the correspondance between the 2 spaces $\mathbb{L}_2(\mathbb{X})$ and $\mathbb{D}_2(\mathbb{N})$ as illustrated below. What can we deduce from all of this? Since the operator acting on the decomposition coefficients converges in distribution to the vector of expansion of the fixed point of the IFS, one can use the statistics of the coefficients to estimate the statistics of the random IFS. This will be achieved in the last part. The choice of the orthonormal basis is therefore crucial and is explained in the following section.

$$\begin{array}{ccc} \mathbb{L}_2(\mathbb{X}) & \xrightarrow{\equiv} & \mathbb{D}_2(\mathbb{N}) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \vartheta} \\ \mathbb{L}_2(\mathbb{X}) & \xrightarrow{\equiv} & \mathbb{D}_2(\mathbb{N}) \end{array}$$

3. Discrete wavelet decomposition

Though not strictly self similar, the wavelet basis is obtained by translations and dilatations of a single mother wavelet, like fixed point of IFS. The two constructions have therefore strong similarities and it is natural to consider wavelet expansion of fractal functions. The discrete wavelet transform represents a signal in terms of a low-pass scaling function ϕ_{00} and a band-pass wavelet function ψ_{00} which can be derived from ϕ_{00}. By considering contractions and dilations of a compactly supported mother wavelet function, $\psi_{ij}(t) = 2^{\frac{i}{2}}\psi_{00}(2^i t - j)$, the family $(\psi_{ij})_{i,j}$ is an $L_2(\mathbb{X})$ orthonormal basis. Any signal with compact support $\mathbb{X} = [0,1]$ can be represented in the following way:

$$f(t) = b_{00}\phi_{00}(t) + \sum_{i\geqslant 0}\sum_{j=0}^{2^i-1} f_{ij}\psi_{ij}(t) \tag{3.1}$$

Without loss of generality, we have considered a decomposition of the signal up to scale 0, reducing the number of scaling coefficients to 1. We now derive the wavelet expansion for an IFS with two maps but the result can be easily extended to IFS with M maps using an M-band wavelet transform. We denote by $W_f(n,m)$ the wavelet coefficient of f at scale n and position m:

$$W_f(n,m) = \int_{[0,1]} \psi_{nm}(t)f(t)dt = \sum_{i=1}^{2}\int_{\omega_i([0,1])} 2^{\frac{n}{2}}\psi(2^n t - m)\phi_i[f^i(\omega_i^{-1}(t)), \omega_i^{-1}(t)]dt$$

Choosing uniform partition maps $\omega_i(t) = \frac{t}{2} + \frac{i-1}{2}$ for $i = 1, 2$, we get:

$$W_f(n,m) = \frac{1}{\sqrt{2}}\sum_{i=1}^{2} W_{\phi_i}\Big(n-1, m - 2^{n-1}(i-1)\Big) \text{ where } n \geqslant 1 \tag{3.2}$$

where $W_{\phi_i}(n,m) = \int_{[0,1]}\psi_{nm}(t)\phi_i(f(t),t)dt$ and whenever $2^{n-1}(i-1) \leqslant m \leqslant i2^{n-1} - 1$, $i = 1, 2$. This clearly emphasizes the redundancy in the wavelet decomposition of fractal attractors of IFS: one can recursively compute the wavelet coefficients in terms of the IFS parameters from coarse to fine scale. When decomposing the signal on a Haar basis, recursive formulae can be computed in the same way for the coarsest coefficient f_{00} and the scaling coefficient b_{00}.

4. Parameter estimation

The inverse problem in the random case is the following. Suppose we know the deterministic parameters of the IFS (contraction ratio, partition maps...) , the goal is then to estimate the statistics (for example the moments) of the randomness introduced in the maps from one or more snapshots of the fixed point. This seems to be impossible in general cases since we cannot infer anything about a random variable just from one realisation, but the scenario is quite different here: by exploiting the fractal property of the fixed point, one has access not to one realisation but to several realisations of the fixed point. This is what we will be using here, in the wavelet domain. In order to derive an estimator, we need to set a particular form for ϕ_i: $\phi_i(u, v) = s_i u + X\zeta_i(v)$ where X is a zero mean Gaussian random variable with variance σ^2. The only parameter to estimate here is the variance of the random variable X. Using (3.2), it is straightforward to derive recursive expression for the variances of each wavelet coefficient:

$$\mathrm{Var} f_{i,j} = \frac{s_1^2}{2}\mathrm{Var} f_{i-1,j} + \frac{\sigma^2}{2}(\zeta^1_{i-1,j})^2 \tag{4.1}$$

$$\mathrm{Var} f_{i,j} = \frac{s_2^2}{2}\mathrm{Var} f_{i-1,j-2^{i-1}} + \frac{\sigma^2}{2}(\zeta^2_{i-1,j-2^{i-1}})^2 \tag{4.2}$$

where ζ^n_{ij} is the wavelet coefficient of ζ_n at scale i and position j. The first equality is valid if $j \in [0, 2^{i-1} - 1]$ and the second one for $j \in [2^{i-1}, 2^i - 1]$. The knowledge of marginal probability density functions of the wavelet coefficients is not enough to derive an estimator of the variance as the coefficients are correlated. One can proceed in the same way as for the variance and using various independencies one can express correlation of any two wavelet coefficients in terms of the IFS parameters. An important fact is that the covariance matrix Γ of the wavelet decomposition can be expressed by $\sigma^2\Upsilon$ where σ^2 is the random parameter to be estimated. One can use the following empirical estimator (here equal as the Maximum Likelihood estimator since we are in the Gaussian case) to estimate the variance of X:

$$\hat{\sigma^2_{ML}} = \frac{1}{N}\mathbf{f}^T\Upsilon^{-1}\mathbf{f} \tag{4.3}$$

where N is the total number of coefficients.

5. Conclusion

We have shown that the wavelet decompostion of the attractor of the IFS acting over the space of functions possesses redundancy from one scale to another, which is little surprise considering the similarities between the wavelet and IFS constructions. This redundancy can be exploited for example to estimate the statistics of random fixed points. It would be interesting to obtain more general estimators in a more general setting, and to adapt the results to 2-D images.

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