NUMERICAL RECONSRUCTION OF SIGNALS AND ORTHOGONAL POLYNOMIALS

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Abstract. Given only a few of initial Fourier coefficients for a continuous-time periodic signal, we construct efficient rational approximants to the whole signal everywhere on his domain of definition. The convergence of these approximants depends on the orthonormality of the chosen generating polynomial system $\{V_{m+1}(e^{it}): m = 0, 1, \dots\}$ into $L^2[-\pi, \pi]$. The form of each $V_{m+1}(x)$ is characterized by recurrence relations dues to the connection between Schur and Szegö theories.

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INTRODUCTION

Given a signal Fourier analysis easily calculates the frequencies and the amplitudes of those frequencies which make up the signal. However, Fourier methods are not always a good tool to recapture the signal, particularly if it is highly non-smooth. Especially, there is no analytical way to reconstruct with exactitude a signal if only a few of its initial Fourier coefficients are known.

In this paper, we will consider *a numerical version of the Carathéodory-Fejér interpolation problem* ([4, 5], [13, 14]) *in the trigonometric context. In particular,* ***we will investigate a numerical method for constructing efficient approximants to any continuous-time periodic signal by using only a few of its initial Fourier coefficients***. These approximants are real parts of rational functions with numerators determined by the condition that the Fourier series expansion of the approximants matches the Fourier series of the signal as far as possible. Motivated by this crucial property, the obtained approximants will be called Padé-type approximants. The convergence of these approximants depends strongly on the orthonormality of the chosen generating polynomial system $\{V_{m+1}(e^{it}): m = 0, 1, \dots\}$ into $L^2[-\pi, \pi]$. The form of $V_{m+1}(x)$ is characterized by recurrence relations dues to the connection between Schur and Szegö theories.

The detailed definition of a Padé-type approximant to a continuous-time periodic L^1- signal and their principal properties are presented in Section 1. Section 2 is devoted to a study of the convergence of a sequence of such approximants, with emphasis on the assumptions under which, for every sequence of functions converging to a periodic continuous signal there is a Padé-type approximation sequence converging point wise to that signal faster than the first sequence. Finally, in Section 3 numerical examples are given making use of Padé-type approximants to signals.

1. CONSTRUCTION OF RATIONAL APPROXIMANTS TO A PERIODIC CONTINUOUS TIME SIGNAL

Consider a $T-$periodic continuous time signal $f(x)$ defined over an interval $[-(T/2), (T/2)]$. Suppose $f \in L^1\big(-(T/2), (T/2)\big)$ and f has a finite number of extrema and discontinuities in any given interval. Then, f has a Fourier series expansion defined by $f(x) = \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu\frac{2\pi}{T}x}$ $(\omega := 2\pi/T)$. The problem we will investigate is the following:

> **if only a few Fourier coefficients $c_0, c_{\pm1}, \dots, c_{\pm\mu}$ of the signal $f(x)$ are given, construct efficient rational approximants to the whole signal $f(x)$ (almost) everywhere on $[-(T/2), (T/2)]$.**

Putting $t := (2\pi/T)x$, the signal $f(x)$ converts to a $2\pi-$periodic continuous time signal $\mathfrak{f}(t)$ defined over the standard interval $[-\pi, \pi]$, with Fourier series expansion $\mathfrak{f}(t) = \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu t}$. It is clear that $\mathfrak{f}$ can be identified with a real-valued function $u(z)$ in L^1 of the unit circle C by setting $u(e^{it}) := \not{f}(t) = \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu t}$. From the solution of the Dirichlet problem in the unit disk D, it follows that the extended real-valued function $u(z) := u(re^{it})$ is harmonic in the open unit disk and such that $lim_{r\to 1}\|u_r(t) - u(e^{it})\|_1 \left(:= lim_{r\to 1}\int_{-\pi}^{\pi}\left[u_r(t) - u(e^{it})\right]dt\right) = 0$ and the Fourier series expansion of the restriction $u_r(t)$ of $u(re^{it})$ to any circle C_r of radius $r < 1$ is given by $\sum_{\nu=-\infty}^{\infty} c_\nu r^{|\nu|} e^{i\nu t}$.

We can consider Padé-type approximants to the harmonic function $u(z)$. Their fundamental property is that the Fourier series expansion of their restriction to any circle of radius $r < 1$ coincides with the Fourier series expansion of $u(z)$ as far as possible. Indeed, $u(z)$ is the real part of an analytic function $F(z)$ in D. So, $u = F + \overline{F}$ where $\overline{F}$ denotes complex conjugate of F. If $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ is the Taylor power series expansion of F around $0 \in D$, then $u(z) = 2Re\left(F(z)\right) - a_0 = 2\ Re(\sum_{\nu=0}^{\infty} a_\nu z^\nu) - a_0$ and $a_\nu = c_\nu$ ($\nu = 0, 1, \dots$). Define the $\mathbb{C}$ −linear functional $T_{\not{f}}: \mathbb{P}(\mathbb{C}) \to \mathbb{C}; x^\nu \mapsto T_{\not{f}}(x^\nu) := a_\nu (= c_\nu)$, where $\mathbb{P}(\mathbb{C})$ is the vector space of all complex analytic polynomials. An application of Cauchy's integral formula shows that $\left|T_{\not{f}}\left(p(x)\right)\right| \le (2\pi)^{-1} sup_{|s|=r}|F(s)| \quad sup_{|s|=r^{-1}}|p(s)|$ whenever $p(x) \in \mathbb{P}(\mathbb{C})$. By density, there is a continuous extension of $T_{\not{f}}$ into the space $\mathcal{O}(\overline{D})$ of all functions which are analytic in an open neighborhood of $\overline{D}$. In particular, for every fixed point $z \in D$, the number $T_{\not{f}}((1 - xz)^{-1})$ is well defined and equals $\sum_{\nu=0}^{\infty} a_\nu z^\nu$. Hence, it holds $u(z) = 2Re\left(T_{\not{f}}((1 - xz)^{-1})\right) - c_0$ for any $z \in D$. If the function $(1 - xz)^{-1}$ is replaced by a polynomial $Q(x, z)$, then $u(z)$ is approximated by $2ReT_{\not{f}}\left(Q(x, z)\right) - c_0$. This is an approximate quadrature formula and leads to a Padé-type approximant to the harmonic real-valued function $u(z)$.

DEFINITION 1.1 ([9, 10]) *Let $Q_m(x, z)$ denote the unique complex polynomial of degree at most m in x, which interpolates the Cauchy kernel $(1 - xz)^{-1}$ at $m + 1$ points $\pi_0, \pi_1, \dots, \pi_m$, i.e., $Q_m(\pi_k, z) = (1 - \pi_k z)^{-1}$ for any $z \in \mathbb{C} \setminus \{\pi_k^{-1};\ k = 0, 1, \dots, m\}$ and $k \le m$. The real-valued function $(m/m + 1)_u(z) := 2ReT_{\not{f}}\left(Q_m(x, z)\right) - c_0$ is said to be a Padé-type approximant to $u(z)$, with generating polynomial $V_{m+1}(x) = \gamma \prod_{k=0}^{m}(x - \pi_k)$ ($\gamma \in \mathbb{C} \setminus \{0\}$).* ■

The approximant in Definition 1.1 can be interpreted as real part of a rational function. Indeed, we put $\widetilde{V}_{m+1}(z) := z^{m+1}\ V_{m+1}(z^{-1})$, $W_m(z) := \ T_{\not{f}}([\ V_{m+1}(x) -\ V_{m+1}(z)]/[x - z])$ and $\widetilde{W}_m(z) := z^m\ W_m(z^{-1})$.

THEOREM 1.2 ([8, 9]) *$(m/m + 1)_u(z)$ is the real part of a rational complex function of type $(m, m + 1$: mm+1uz=2Re Wmz Vm+1z−c0. The error of the respective approximation equals*

$$(m/m + 1)_u(z) - u(z) = 2Re\left[\frac{1}{V_{m+1}(z^{-1})} T_{\not{f}}\left(\frac{V_{m+1}(x)}{xz - 1}\right)\right].$$

The fundamental property of these approximants is the following. The Fourier series expansion $\sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} r^{|\nu|} e^{i\nu t}$ of the restriction $(m/m + 1)_{u_r}(t)$ of $(m/m + 1)_u(re^{it})$ to the circle of radius r fulfills $d_\nu^{(m)} = c_\nu$ for any $\nu = 0, \pm 1, \pm 2, \dots, \pm m$. ■

Since the limit $lim_{r\to 1}(m/m + 1)_{u_r}(t)$ is uniform on $[-\pi, \pi]$, the function $(m/m + 1)_u(re^{it})$ is the Poisson integral of a continuous function on the unit circle.

DEFINITION 1.3 ([8, 9]) *The radial limit*

$$(m/m + 1)_{\not{f}}(t) := lim_{r\to 1}(m/m + 1)_{u_r}(t) = 2ReT_{\not{f}}\left(Q_m(x, e^{it})\right) - c_0$$

is said to be a Padé-type approximant to $\not{f}(t)$, with generating polynomial

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m}(x - \pi_k)\ (\ \gamma \in \mathbb{C} \setminus \{0\}). \ ■$$

By Theorem 1.2, we get the following properties of these approximants.

THEOREM 1.4 *The Padé-type approximant $(m/m + 1)_{\not{f}}(t)$ is the real part of a rational complex function of type $(m, m + 1)$: $(m/m + 1)_{\not{f}}(t) = 2Re(\ \widetilde{W}_m(e^{it}) / \ \widetilde{V}_{m+1}(e^{it})) - c_0$.*
The error of the respective approximation is given by the following theoretical formula

$$(m/m + 1)_{\not{f}}(t) - \not{f}(t) = \frac{1}{\pi} lim_{r\to 1} Re\left\{\frac{1}{V_{m+1}(r^{-1}e^{-it})}\int_{-\pi}^{\pi}\frac{\not{f}(s)\ V_{m+1}(e^{-is})}{[re^{i(t-s)} - 1]} ds\right\}$$

where the limit is taken in the L^1–norm. The Fourier series expansion $\sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} e^{i\nu t}$ of $(m/m + 1)_{\not{f}}(t)$ fulfills $d_\nu^{(m)} = c_\nu$ for any $\nu = 0, \pm 1, \pm 2, \dots, \pm m$.

Proof We will use the standard identification of the $2\pi-$ periodic signal $f(t)$ $(t\in[-\pi,\pi])$ with the L^1 –function $u(z)\equiv u(e^{it})$ $(z=e^{it},\ |z|=\ 1)$. Let $(r_n\in]0,1[)_{n=0,1,2,\dots}$ be any sequence such that $lim_{n\to+\infty}r_n=1$. Since, by Theorem 1.2, $(m/m+1)_u(r_ne^{it})=2Re(\ \widetilde{W}_m(r_ne^{it})/\ \widetilde{V}_{m+1}(r_ne^{it}))-c_0$, the uniform convergence of the sequence $\left((m/m+1)_u(r_ne^{it})\right)_{n=0,1,2,\dots}$ to the radial limit function $(m/m+1)_u(e^{it})$ implies that $(m/m+1)_u(e^{it})=2Re(\ \widetilde{W}_m(e^{it})/\ \widetilde{V}_{m+1}(e^{it}))-c_0$, and the first assertion is proved. To prove the second assertion, recall that $lim_{n\to+\infty}\left\|u(r_ne^{it})-u(e^{it})\right\|_1=0$. Further, by the uniform convergence of the sequence $\left((m/m+1)_u(r_ne^{it})\right)_{n=0,1,2,\dots}$ to the radial limit function $(m/m+1)_u(e^{it})$ we also get $lim_{n\to+\infty}\left\|(m/m+1)_u(r_ne^{it})-\ (m/m+1)_u(e^{it})\right\|_1=0$. Letting $\varepsilon>0$, from Theorem 1.2, it follows that there exists a $N=N(\varepsilon)$ such that

$$\left\|\left[(m/m+1)_u(e^{it})-u(e^{it})\right]-2Re\left[\frac{1}{V_{m+1}(r_n^{-1}e^{-it})}T_f\left(\frac{V_{m+1}(x)}{xr_ne^{it}-1}\right)\right]\right\|_1<\varepsilon$$

for any $n\geq N$. Now, set $V_{m+1}(x)=\sum_{k=0}^{m+1}b_k^{(m)}\,x^k$. An application of the continuity property for the linear functional T_f shows that

$$T_f\left(\frac{V_{m+1}(x)}{xr_ne^{it}-1}\right)==\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{is})\left[\frac{V_{m+1}(e^{-is})}{r_ne^{i(t-s)}-1}\right]ds=\int_{-\pi}^{\pi}\frac{f(s)\ V_{m+1}(e^{-is})}{r_ne^{i(t-s)}-1}ds.$$

Hence

$$\left\|\left[(m/m+1)_f(t)\ -f(t)\right]-2Re\left[\frac{1}{V_{m+1}(r_n^{-1}e^{-it})}\int_{-\pi}^{\pi}\frac{f(s)\ V_{m+1}(e^{-is})}{r_ne^{i(t-s)}-1}ds\right]\right\|_1<\varepsilon,$$

for any $n\geq N$, which proves the second assertion. It remains to prove the third assertion. Since every harmonic function in the unit disk, with continuous boundary values, is the Poisson integral of its continuous restriction to the unit circle, we have

$$(m/m+1)_u(re^{it})=\sum_{\nu=-\infty}^{\infty}r^{|\nu|}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}(m/m+1)_u(e^{i\theta})\ e^{i\nu\theta}\ d\theta\right]e^{it}.$$

Henceforth, the Fourier series expansion of $(m/m+1)_u(re^{it})$ is $\sum_{\nu=-\infty}^{\infty}d_\nu^{(m)}r^{|\nu|}\,e^{it}$. From Theorem 1.2, it follows that $d_\nu^{(m)}=c_\nu$ for any $\nu=0,\pm1,\pm2,\dots,\pm m$, and the proof is complete. ■

From Theorem 1.4, it follows immediately that **if only a few Fourier coefficients c_ν of the signal $f(t)$ are given, then one can approximate f by an approximant in the Padé-type**: $(m/m+1)_f(t)\approx f(t)$.

2. CONVERGENCE ACCELERATION AND ORTHOGONAL POLYNOMIALS

In this Section, we shall study the convergence of a sequence of Padé-type approximants to a signal. Let $\mathcal{M}=(\pi_{m,k})_{m=0,1,\dots;k=0,1,\dots,m}$ be an infinite triangular interpolation matrix with complex entries $\pi_{m,k}\in D$. For any fixed $z\in\mathbb{C}\setminus\{\pi_{m,k}^{-1};\ m=0,1,\dots\ and\ \ k=0,1,\dots,m\}$, let $Q_m(x,z)$ denote the unique polynomial of degree at most m in x, which interpolates the Cauchy kernel $(1-xz)^{-1}$ in the $(m+1)$ nodes of the m^{th} row of $\mathcal{M}$, i.e., $Q_m(\pi_{m,k},z)=(1-\pi_{m,k}z)^{-1}$ for any $k\leq m$.

THEOREM 2.1 *Suppose the family* $\{V_{m+1}(e^{it}):m=0,1,\dots\}$ *of generating polynomials is an orthonormal bounded system in* $L^2[-\pi,\pi]$ *and* $inf_{m\geq M_0}inf_{t\in[-\pi,\pi]}\left|V_{m+1}(e^{it})\right|>0$, *for a* M_0. *Then for any real-valued* 2π *–periodic continuous-time signal* $f(t)\in L^1[-\pi,\pi]$, *the associated sequence* $\left((m/m+1)_f(t)=2ReT_f\left(Q_m(x,e^{it})\right)-c_0\right)_{m=0,1,\dots}$ *of Padé-type approximants to* $f(t)$, *with generating polynomials* $V_{m+1}(x)=\prod_{k=0}^{m}(x-\pi_{m,k})$, *converges to* $f(t)$ *almost everywhere on* $[-\pi,\pi]$. *Especially, if* $f(t)\in C[-\pi,\pi]$, *the sequence* $\left((m/m+1)_f(t)=2Re(\ \widetilde{W}_m(e^{it})/\ \widetilde{V}_{m+1}(e^{it}))-c_0\right)_{m=0,1,\dots}$ *converges to* $f(t)$ *everywhere on* $[-\pi,\pi]$.

Proof Let $f(t)\in L^1[-\pi,\pi]$ be a real-valued 2π –periodic continuous-time signal. Let also $\varepsilon>0$ and $(r_n)_{n=0,1,2,\dots}$ be a strictly increasing sequence of positive numbers such that $lim_{n\to\infty}r_n=1$. Fix any n. By Theorem 1.4, there is a subsequence $\left(r_{n_j}\right)_{j=0,1,2,\dots}$ of (r_n) such that

$$(m/m+1)_f(t)-f(t)=\frac{1}{\pi}lim_{j\to\infty}Re\left\{\frac{1}{V_{m+1}\left(r_{n_j}^{-1}e^{-it}\right)}\int_{-\pi}^{\pi}\frac{f(s)\ V_{m+1}(e^{-is})}{r_{n_j}e^{i(t-s)}-1}ds\right\}$$

for almost all $t\in[-\pi,\pi]$. Denote by $\mathfrak{D}$ the set of all points $t\in[-\pi,\pi]$ with this property. For every $t\in\mathfrak{D}$, one can find a $J=J(\varepsilon,m)$ such that

- $\left|(m/m+1)_f(t)-f(t)\right|\leq\frac{1}{\pi}\left|\frac{1}{V_{m+1}\left(r_{n_j}^{-1}e^{-it}\right)}\right|\left|\int_{-\pi}^{\pi}\frac{f(s)\ V_{m+1}(e^{-is})}{r_{n_j}e^{i(t-s)}-1}ds\right|+\frac{\varepsilon}{2}\,(j\geq J)$ and

- the function $[-\pi,\pi] \to \mathbb{C}: s \mapsto \left(f(s)/\left[r_{n_j} e^{i(t-s)} - 1\right]\right)$ is in $L^1[-\pi,\pi]$, so, by Mercer's Theorem, the Fourier coefficients of this function with respect to the orthonormal family $\{V_{m+1}(e^{-is}): m = 0,1,\dots$ tend to zero. This means that there exists a $M=M\varepsilon, t$ such that

$$\frac{1}{\pi}\left|\frac{1}{V_{m+1}\left(r_{n_j}^{-1}e^{-it}\right)}\right|\left|\int_{-\pi}^{\pi}\frac{f(s)\ V_{m+1}(e^{-is})}{r_{n_j}e^{i(t-s)}-1}ds\right| < \frac{\varepsilon}{2} \text{ (for any } m \geq M \text{ and } j \geq J).$$

Combination of these inequalities shows that $|(m/m+1)_f(t) - f(t)| < \varepsilon$ for any $m \geq M$. This implies that $lim_{m\to\infty}(m/m+1)_f(t) = f(t)$ for almost all $t \in [-\pi,\pi]$. If $f(t) \in C[-\pi,\pi]$, the convergence holds for any $t \in [-\pi,\pi]$. ■

COROLLARY 2.2 *Let the generating polynomials of a Padé-type approximation $V_{m+1}(x) = \sum_{k=0}^{m+1} b_k^{(m)} x^k = \prod_{k=0}^{m}(x - \pi_{m,k})$ $(m = 0, 1, \dots)$ be such that*

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|^2 = 1/2\pi\ (m \geq 0) \text{ and } \sum_{k=0}^{m+1} b_k^{(m)}\ \overline{b_k^{(n)}} = 0\ (n \geq m).$$

If there are two constants $\sigma < \infty$ and $\tau < 1$ fulfilling

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right| < \sigma\ \ (m \geq 0) \text{ and } |\pi_{m,k}| < \tau\ (m \geq 0 \text{ and } 0 \leq k \leq m).$$

Then for any real-valued 2π −periodic continuous-time signal $f(t) \in L^1[-\pi,\pi]$, the associated sequence $\left((m/m+1)_f(t) = 2Re\left(\ \widetilde{W}_m(e^{it})/\ \widetilde{V}_{m+1}(e^{it})\right) - c_0\right)_{m=0,1,\dots}$ of Padé-type approximants to $f(t)$, with generating polynomials $V_{m+1}(x) = \prod_{k=0}^{m}(x - \pi_{m,k})$, converges to $f(t)$ almost everywhere on $[-\pi,\pi]$. Especially, if $f(t) \in C[-\pi,\pi]$, the sequence $\left((m/m+1)_f(t) = 2Re\left(\ \widetilde{W}_m(e^{it})/\ \widetilde{V}_{m+1}(e^{it})\right) - c0m=0,1,\dots\right.$ converges to ft everywhere on $-\pi,\pi$. ■

In [10], we gave a stronger sufficient condition in terms of the entries $\pi_{m,k}$ only. *If the interpolation points $\pi_{m,k}$ are chosen so that $-1 < \pi_{m,k} < 1$ and $lim_{m\to\infty}\sum_{n>1}\frac{1}{n}\sum_{k=0}^{m}(\pi_{m,k})^n = -\infty$, then, for any real-valued 2π −periodic continuous-time signal $f(t) \in C[-\pi,\pi]$, the associated sequence $\left((m/m+1)_f(t)\right)_{m=0,1,\dots}$ of Padé-type approximants to $f(t)$, with generating polynomials $V_{m+1}(x) = \prod_{k=0}^{m}(x - \pi_{m,k})$, converges to $f(t)$ everywhere on $[-\pi,\pi]$.*

One can also obtain interesting results about the form of $V_{m+1}(x)$, dues to the connection between Schur and Szegö theories. This connection is often attributed to Akhiezer ([1]), but it appears earlier and in greater detail in the papers of Geronimo ([11, 12]). It based on important recurrence relations which were first given in previous Szegö's work ([15]). *Denoting by $V_{m+1}^*(x)$ the polynomial $x^{m+1}\overline{V_{m+1}(1/\overline{x})}$, the recurrence relations written in terms of the monic polynomials $V_{m+1}(x) = V_{m+1}(x)/V_{m+1}^*(0)$ are of the general form $V_{m+2}(x) = x\,V_{m+1}(x) - \overline{a}_{m+1}V_{m+1}^*(x)$ for certain parameters $a_{m+1} \in \mathbb{C}$ $(m = 0, 1, \dots)$.* In current terminology the numbers $-\overline{a}_{m+1} = V_{m+2}(0)$ are called *Szegö parameters*. Let us see what the Schur parameters are. To do so, we may remind the following Schur's construction.

THEOREM 2.3 ([13, 14])*There is a one-to-one correspondence between the class $S(D)$ of analytic functions which are bounded by one on the unit disk D and the set of all sequences $(\gamma_{m+1})_{m=0,1,2,\dots}$ of complex numbers which are bounded by 1 and such that if some term has unit modulus, then all subsequent terms are zero.*

Proof Given any $\phi(x) \in S(D)$, define a sequence

$$\phi_{m+1}(x) = \begin{cases} \phi(x), if\ m = 0 \\ [\phi_m(x) - \phi_m(0)]/x\left[1 - \overline{\phi_m(0)}\phi_m(x)\right], if\ m \geq 1. \end{cases}$$

If $|\phi_i(0)| = 1$ for some i, then $\phi_i(x)$ is constant and we taken $\phi_j(x) = 0$ for any $j > i$. This occurs if and only if $\phi(x)$ is finite Blaschke product of i factors:

$$\phi(x) = c\left([x - b_1]/[1 - \overline{b_1}x]\right)\left([x - b_2]/[1 - \overline{b_2}x]\right)\cdots\left([x - b_i]/[1 - \overline{b_i}x]\right),$$

where $|c| = 1$ and b_1, b_2,..., b_i are points in D. The numbers $\gamma_{m+1} = \gamma_{m+1}^{(\phi)} := \phi_{m+1}(0)$ $(m \geq 0)$ are defined to be the *Schur parameters for $\phi(x)$*. ■

Since $g(x) = (2\pi)^{-1}\int_{-\pi}^{\pi}([e^{is} + x]/[e^{is} - x])\,ds$ has positive real part in the open unit disk D and value 1 at the origin, the function $\Phi(x) = (1/x)([g(x) - 1]/[g(x) + 1])$ belongs to $S(D)$. In 1943, Geronimo showed that *the Shur parameters $\gamma_{m+1} = \gamma_{m+1}^{(\Phi)}$ for $\Phi(x)$ coincide with the numbers a_{m+1}: $\gamma_{m+1} = \gamma_{m+1}^{(\Phi)} = a_{m+1}$* whenever $m = 0, 1, \dots$ ([15]).

We shall now study assumptions under which, **for every sequence of signals converging to a continuous 2π −periodic signal, there is always a Padé-type approximation sequence converging pointwise to the signal faster than the sequence**. This property, due to the free choice of the interpolation points $\pi_{m,k}$ permits us to construct better and better approximations to continuous functions.

Using techniques similar to those proposed by Bromwich and Clark ([3, 6]), we can prove the

PROPOSITION 2.4 *Let Δ be the operator of differences. Let also $(x_m)_{m=0,1,2,\ldots}$ and $(y_m)_{m=0,1,2,\ldots}$ be two convergent sequences of real numbers. Suppose $(y_m)_{m=0,1,2,\ldots}$ is strictly monotone and $lim_{m\to\infty} y_m = 0$. If $\overline{lim}_{m\to\infty}|\Delta x_m|^{1/m} < lim_{m\to\infty}|\Delta y_m|^{1/m} = 0$, the sequence $(x_m)_{m=0,1,2,\ldots}$ converges faster than the sequence $(y_m)_{m=0,1,2,\ldots}$.* ■

Combination of Proposition 2.4 with Theorem 2.1 gives the following result.

THEOREM 2.5 *Let $(y_m)_{m=0,1,2,\ldots}$ be any strictly monotone converging sequence. Suppose the family $\{V_{m+1}(e^{it}): m = 0, 1, \ldots\}$ is an orthonormal bounded system in $L^2[-\pi,\pi]$ and $inf_{m\geq M_0} inf_{t\in[-\pi,\pi]}|V_{m+1}(e^{it})| > 0$, for a M_0. Then, for any real-valued 2π −periodic continuous-time signal $f(t)$, the associated sequence $\left((m/m+1)_f(t) = 2Re\left(\widetilde{W}_m(e^{it})/\widetilde{V}_{m+1}(e^{it})\right) - c_0\right)_{m=0,1,\ldots}$ of Padé-type approximants converges to $f(t)$ faster than $(y_m)_{m=0,1,2,\ldots}$.everywhere on*

$$\left\{t \in [-\pi,\pi]: lim_{m\to\infty}|\Delta y_m|^{\frac{1}{m}} \geq \overline{lim}_{m\to\infty}\left[sup_{|x|\leq 1}\left|\frac{V_{m+1}(x)\,V_{m+2}(e^{-it}) - V_{m+1}(e^{-it})\,V_{m+2}(x)}{1-xe^{it}}\right|\right]^{\frac{1}{m}}\right\}. \blacksquare$$

3. NUMERICAL EXAMPLES

In this section, we give examples making use of Padé-type approximants to 2π −periodic continuous-time signals $f(t)$.

EXAMPLE 4.1 Let f be the signal $f(t) = |t|$ $(t \in \mathbb{R})$. If $m = 4$ and $\pi_{4,k}$ are the roots of the fifth Legendre polynomial in $[-1, 1]$, then $V_5(x) = \left(63x^5 - 70x^3 + 15x\right)/8$ and therefore

$$(4/5)_f(t) = \frac{9094\pi - \frac{5292}{\pi}cost - 10920\pi\, cos2t + \frac{12096}{\pi}cos3t + 1890\pi\, cos4t}{(63-70\, cos2t + 15\, cos4t)^2 + (-70 sin2t + 15 sin4t)^2} - \frac{\pi}{2}$$

$(t \in [-\pi,\pi])$. Indicatively, we have $(4/5)_f(\pm\pi/2) = 1.570796 = f(\pm\pi/2)$ and $(4/5)_f(\pm 1) = 1.0957434 \approx 1 = f(\pm 1)$. If $m = 4$ and $\pi_{4,k} = 0$, then $V_5(x) = x^5$ and therefore $(4/5)_f(t) = (\pi/2) - (4cost)/\pi - (4cos3t)/9\pi$ $(t \in [-\pi,\pi])$. Indicatively, we have $(4/5)_f(0) = 0 = f(0)$, $(4/5)_f(\pm\pi/2) = 1.5707963 = f(\pm\pi/2)$, $(4/5)_f(\pm\pi/8) = 0.392699 \approx 0.3403377 = f(\pm\pi/8)$, $(4/5)_f(\pm 1) = 1.0756475 \approx 1 = f(\pm 1)$ and $(4/5)_f(\pm e) = 2.7182818 \approx 2.673454 = f(\pm e)$.

EXAMPLE 4.2 Let f be the signal $f(t) = [r\, sint]/[1 - 2r\, cost + r^2]$ $(t \in \mathbb{R})$. If $m = 3$ and $\pi_{3,k} = cos[(2k+1)\pi/7]$, then $V_4(x) = x^4 + 0.5x^3 - x^2 - 0.375x + 0.125$ and therefore

$$\begin{aligned}(3/4)_f(t) = -r \times \big([&1.125r^2 - 0.3125r - 1.0625]sint + \\ &[-0.5r^2 - 1.125r - 1.3125]sin2t + [-r^2 + 0.5r + 1.375]sin3t\big) \\ &\times \big([1 + 0.5cost - cos2t - 0.375cos3t + 0.125cos4t]^2 \\ &\qquad + [0.5sint - sin2t - 0.375sin3t + 0.125sin4t]^2\big)^{-1}\ (t \in [-\pi,\pi]).\end{aligned}$$

In particular, it holds

- $(3/4)_f(\pi/3) = [0.8660254\, r(3.413294 + r)(1.113294 - r)/4.5468749] \approx$ $[0.8660254\, r/(1 - r - r^2)] = f(\pi/3)$,
- $(3/4)_f(\pi/4) = r\,(r - 0.018477)(11.20945 - r)/2.376569 \approx$ $0.7071067\, r/(1 - 1.4142135r - r^2) = f(\pi/4)$,
- $(3/4)_f(0) = 0 = f(0)$,
- $(3/4)_f(-\pi) = 0 = f(-\pi)$,
- $(3/4)_f(\pi/5) = -r\left(0.7653264r^2 + 0.7775396r + 0.5654351\right) / 2.2188258 \approx$ $0.5877852\, r/(1 - 1.6180339r + r^2) = f(\pi/5)$ and
- $(3/4)_f(-\pi/6) = [-r(0.8705127r^2 + 0.6305285r + 0.2929083)/1.5370791] \approx$ $-0.5\, r/(1 - 1.7320508r + r^2) = f(-\pi/6)$.

If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0$ and $\pi_{3,3} = -i$, then $V_4(x) = x^4 + ix^3$ and therefore

$(3/4)_f(t) = \frac{r}{2(1-sint)} \times (-1 + 2sint + r\, sin2t + r^2 sin3t + r\, sin4 - r\, cost + [1 - r^2]cos2t + r\, cos\, 3t)$

($t \in [-\pi, \pi] \setminus \{\pi/2\}$). Observe that $(3/4)_{\mathfrak{f}}(t)$ is well defined everywhere on $[-\pi, \pi]$, with the exception of $t = \pi/2$. This is a consequence of our choice $\pi_{3,3} = -\mathrm{i}$, which in particular implies $|\pi_{3,3}| = 1$. However,

- $(3/4)_{\mathfrak{f}}(0) = -0.5r^3 \approx 0 = \mathfrak{f}(0)$,
- $(3/4)_{\mathfrak{f}}(\pi/3) = 3.7320507\, r\,(0.7320508 - 1.5r + 0.5\,r^2) \approx 0.8660254\, r/(1 - r - r^2) = \mathfrak{f}(\pi/3)$,
- $(3/4)_{\mathfrak{f}}(\pi/4) = 1.7071067\, r(0.4142135 - 0.4142135r + 0.7071067r^2) \approx$ $0.7071067\, r/(1 - 1.4142135r - r^2) = \mathfrak{f}(\pi/4)$ and
- $(3/4)_{\mathfrak{f}}(-\pi) = 0.5\, r\,(1 - r^2) \approx 0 = \mathfrak{f}(-\pi)$.

EXAMPLE 4.3 Let finally $\mathfrak{f}$ be the signal

$$\mathfrak{f}(t) = \begin{cases} 1, & if\ t = -\pi \\ 2\left(\frac{t}{\pi} + 1\right), & if\ -\pi < t \le 0 \\ 2, & if\ 0 \le t < \pi \\ 1, & if\ t = \pi \end{cases}$$

If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0$, $\pi_{3,3} = -1/2$, then $V_4(x) = x^4 + (x^3/2)$ and

- $(3/4)_{\mathfrak{f}}(t) = 4(3.9526423 + 0.6366197\, sint + +3.5066059\, cost + 0.251163\, cos\, 2 - 0.1061032\, sin3t +$ $0.0450316\, cos3t - 0.0506655\, sin4t\,)/(5 + 4cost) - 1.5$ ($t \in [-\pi, \pi]$).

In particular, we have $(4/5)_{\mathfrak{f}}(\pi/2) = 2.0553617 \approx 2 = \mathfrak{f}(\pi/2)$, $(4/5)_{\mathfrak{f}}(\pi/3) = 2.0031648 \approx 2 = \mathfrak{f}(\pi/3)$, $(4/5)_{\mathfrak{f}}(-\pi/3) = 1.3730748 \approx 1.3333333 = \mathfrak{f}(-\pi/3)$, $(4/5)_{\mathfrak{f}}(\pi/5) = 1.9466674 \approx 2 = \mathfrak{f}(\pi/5)$ and $(4/5)_{\mathfrak{f}}(-\pi/5) = 1.5967140 \approx 1.6 = \mathfrak{f}(-\pi/5)$.

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