Don't Switch! Why Mathematicians' Answer to the Monty Hall Problem is Wrong

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The Monty Hall problem is one of those rare curiosities – a mathematical problem that has made the front pages of national news. Everyone now knows, or thinks they know, the answer but a realistic look at the problem demonstrates that the standard mathematician's answer is wrong. The mathematics is fine, of course, but the assumptions are unrealistic in the context in which they are set. In fact, it is not clear that this problem can be appropriately addressed using the standard tools of probability theory and this raises questions about what we think probabilities are and the way we teach them.

1 Introduction

The Monty Hall problem hit the headlines in 1990, when Craig F. Whitaker of Columbia, Maryland, asked Marilyn vos Savant: 'Suppose you're on a game show, and you're given the choice of three doors: behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" ' Is it to your advantage to take the switch?'[1]

Vos Savant wrote a column called 'Ask Marilyn' in the popular magazine *Parade*, in which she responded to readers' questions. According to the *Guinness Book of Records*, at the time she was the woman with the highest IQ in the world.

Vos Savant responded to Whitaker in her column of 9 September 1990: she said you should switch and that you double your chances of winning if you do. The result was a torrent of criticism and abuse – much of it from mathematicians – such as:

- 'May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?' (Charles Reid, PhD, University of Florida)
- 'You blew it, and you blew it big! ... There is enough mathematical illiteracy in this country, and we don't need the world's highest IQ propagating more. Shame!' (Scott Smith, PhD, University of Florida)
- 'You made a mistake, but look at the positive side. If all those PhD's were wrong, the country would be in some very serious trouble.' (Everett Harman, PhD, US Army Research Institute)

She followed it up with another article on 2 December, addressing the problem in more detail, to which there was more criticism and abuse. Vos Savant said she received 10,000 letters about her articles, including almost 1,000 from PhDs. 'Of the letters from the general public, 92% are against my answer, and of the letters from universities, 65% are against my answer. Overall, nine out of ten readers completely disagree with my reply.' In a third article on 17 February 1991, she spelt out her reasoning in even more detail, and suggested school classes carry out empirical trials. These were carried out and supported her view that switching increases your chances of winning, and now most of her correspondents agreed with her. On Sunday, 21 July 1991, John Tierney wrote a front page article in the *New York Times* that was very firmly on her side.[2]

It looks as if America is in some very serious trouble!

There were some complaints that vos Savant's first article did not fully spell out all the assumptions underlying her answer and which constituted a subtle extension of the original question. Nor were her arguments mathematically rigorous. She accepted these points but argued that her respondents had clearly not been confused, and that mathematical rigour had no place in what was essentially a light-hearted magazine column. We will come back to the question of those assumptions later.

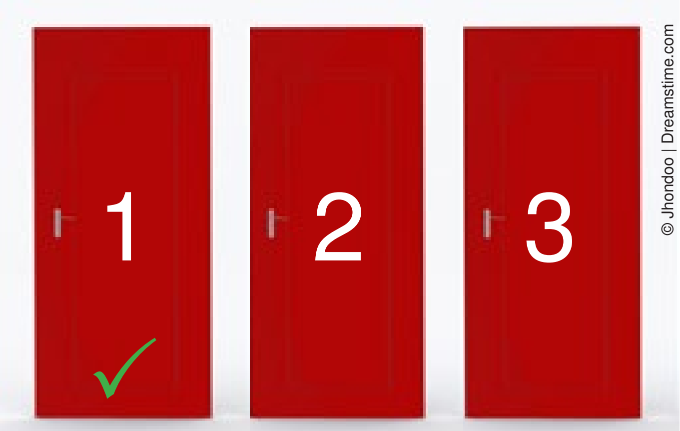

Figure 1: You pick door 1.

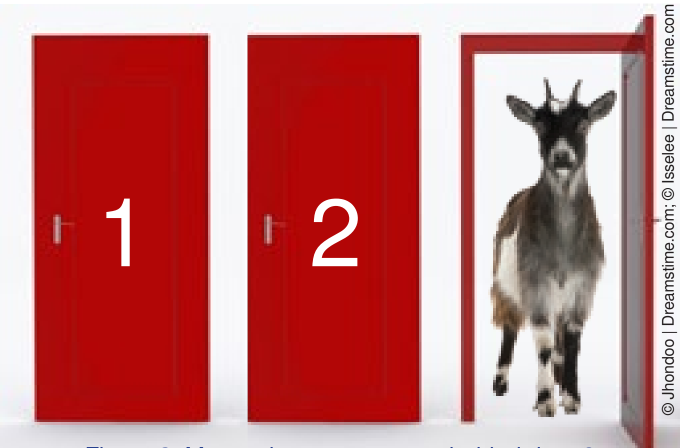

Figure 2: Monty shows you a goat behind door 3.

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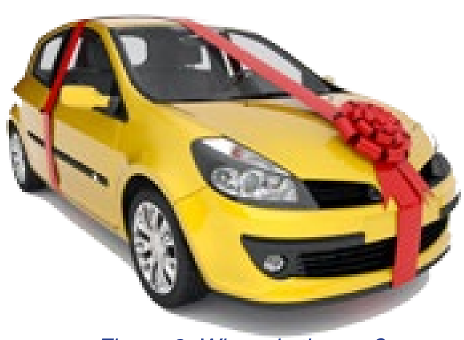

Figure 3: Where is the car?

2 A brief history of the Monty Hall problem

Monty Hall, real name Maurice Halperin (born 1921), was a Canadian TV personality, who hosted the American television game show *Let's Make a Deal* in the 1960s and 1970s. In this programme, Monty offered many different types of challenge to contestants and the Monty Hall problem is supposedly based on one of them, though in fact the game as described above did not appear on the show. The ideas behind the Monty Hall problem were far from new, though. Joseph Bertrand's Box paradox, which he described in 1889 [1], was based on similar ideas and Martin Gardner's Three Prisoners problem of 1959 [2] was equivalent to it in mathematical terms.

The modern version, but without the goats, was described in a letter to *American Statistician* by Professor Steve Selvin of the University of California in 1975 [3]. This introduced Monty Hall (but spelt Monte) and *Let's Make a Deal*. Like vos Savant, Selvin received letters claiming his solution was incorrect so he expanded on his original solution in a second letter later the same year. He noted, in particular, that: 'Benjamin King pointed out the critical assumptions about Monty Hall's behavior that are necessary to solve the problem' [4]. It was in this second letter that the name 'Monty Hall problem' first appeared in print. The goats appear to have originated with Whitaker.

3 The mathematics of the Monty Hall problem

There are various ways to solve the Monty Hall problem mathematically. Here is one using Bayes' theorem, which tells us how to compute certain types of conditional probabilities. It is sometimes called the 'probability of causes' theorem, because if we have an outcome that can arise in different ways ('causes'), it tells us how likely those causes are in the light of the outcome. Despite the fact that it can be deduced relatively easily from the basic rules and definitions of probability theory and is absolutely sound mathematically, its use sometimes gives rather unintuitive results.

The general form of Bayes' theorem is:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_i P(A|B_i)P(B_i)}$$

where A is the result that we have and the B_i are a set of mutually exclusive events that includes all the possible 'causes'. Bayes' theorem has the advantage that, not only does it give the correct answer, it forces us to be explicit about our assumptions.

It deals very easily with the Monty Hall problem. Let C_i be the event 'the car is behind door i' and D_i be the event 'Monty shows you a goat behind door i'. Suppose you choose door 1 and Monty shows you a goat behind door 3. What we need to know is the conditional probability $P(C_1|D_3)$, the probability that the car is behind the door you have chosen, now that Monty has shown us a goat behind door 3.

By Bayes' theorem, assuming that Monty will *always open a door*, *never open the door you have chosen* and *never show us the car* (so the D_i are the only possibilities we need to take into account):

$$P(C_1|D_3) = \frac{P(D_3|C_1)P(C_1)}{P(D_3|C_1)P(C_1) + P(D_3|C_2)P(C_2) + P(D_3|C_3)P(C_3)}.$$

If we assume that *the car is initially equally likely to be behind any door* then the $P(C_i)$ are easy, they are all $1/3$ but the C's and D's are not independent so what are the conditional probabilities? Let us start with $P(D_3|C_1)$. Now door 1 conceals the car so both doors 2 and 3 conceal goats, so *if Monty always chooses at random* $P(D_3|C_1) = 1/2$. And what is $P(D_3|C_2)$? Now our door (door 1) conceals a goat and door 2 conceals the car so Monty has no choice but to open D_3 as he never shows us the car, so $P(D_3|C_2) = 1$. Similarly $P(D_3|C_3) = 0$. Plug these numbers in:

$$P(C_1|D_3) = \frac{1/2 \times 1/3}{1/2 \times 1/3 + 1 \times 1/3 + 0 \times 1/3} = \frac{1}{3}$$

and we see that the probability our original door conceals the car is still $1/3$. The car is not behind door 3 so the probability that it is behind door 2 must be $2/3$. So switch.

4 The assumptions

The mathematics is correct, so you do indeed seem to double your chances by switching but only provided certain assumptions hold. As the words in italics above show, there are actually a number of assumptions:

1. Monty will always open a door.
2. Monty never opens the door you have chosen.
3. Monty never opens the door with the car behind it.
4. The car is equally likely to be behind any door.
5. Given a choice of doors, Monty chooses at random.

Where do these assumptions come from and just how plausible are they?

If we had watched Monty play this game many times we might have been able to spot a pattern that would justify them but, as we have seen, Monty Hall never played this game; no comfort from that source. Where do they come from? They were not in Craig Whitaker's original query given above but vos Savant's added some crucial words in her response: 'the host, who knows what's behind the doors and *will always avoid the one with the prize*' (my emphasis). Vos Savant has done what all academics do when putting this problem to students (and I do when I am putting it to mine), she has turned a real problem that mathematicians cannot answer because they don't know what the true probabilities are into one they can answer, by making plausible assumptions. Except that it is not clear in this case how plausible the assumptions are.

Let us start by thinking about Monty's objectives and motivations. As a game show host it is reasonable to assume that he has a number of things to worry about:

1. He has to *manage* the game; that is, he has to ensure that all the contestants have a reasonable hearing and that it finishes on time.
2. He has to *entertain* the audience; this is likely to mean, among other things, that the contestants win cars reasonably often.
3. He is not likely to want to *give away too many cars* because of the cost to his employers.

He may have other motivations but these will do for now.

In the light of these, how plausible are the assumptions above?

Let us take them one by one:

> **Assumption 1:** That Monty will always open a door. This seems entirely reasonable. It would be a very odd game show where there wasn't a clear outcome at the end.
>
> **Assumption 2:** That Monty never opens the door you have chosen. It is not at all clear why this should be the case. Why should Monty, after the usual banter associated with game shows ('Do you want to switch? Are you sure you don't want to switch?'), not simply open the door you have chosen and tell you whether you have won or not? Indeed, if he is running short of time, if he knows there is a car behind the door and no-one has won for a while or he knows there is a goat behind the door and a number of people have won cars recently, is this not what he is likely to do?
>
> **Assumption 3:** That Monty never opens the door with the car behind it. This assumption is again rather dubious.[3] Why shouldn't Monty simply open a door and show you the car, particularly if he is running out of time or wants to engineer a particular outcome. In practice he is likely to open the door you have chosen rather than the door with the car behind it if you have lost and retain an air of mystery over the location of the car, but the effect is the same.
>
> **Assumption 4:** That the car is equally likely to be behind any door. There is no reason to believe that a particular door is likely to be preferred, so this seems reasonable.
>
> **Assumption 5:** That, given a choice of doors, Monty chooses at random. This assumption does not necessarily hold – Monty may be inherently lazy or have a bad leg and so have a tendency to open the available door nearest to him – but it is easy to show that if all the other assumptions hold you cannot lose by switching, though you don't necessarily gain, so you might as well switch.

The key questionable assumptions, then, are 2 and 3 – that Monty will never open the door you have chosen and will never open the door with the car behind it – and it is these two assumptions that are at odds with our intuitive understanding of the way game shows work. We all know that it is more fun, for the audience at least, if someone is conned out of a winning choice, so there is always a suspicion that that is what Monty is trying to do.

We also know that if a game show is to be entertaining it has to be varied. If Monty behaves too simplistically the game will become predictable and so less entertaining. That leads on to a rather deeper point – it is not at all clear that we can apply probability theory, at least in the traditional sense, to this problem at all. Traditionally, probability theory applies to certain types of repeatable event where the outcomes are random (in a sense we need not dwell on here) but *Monty Hall is not a random event*. He is not a coin to be tossed or a die to be thrown; he is a rational being with free will (or at least the will of his producers) and certain objectives to achieve. Because a primary objective is to entertain, his choices will not be random in the way that simple probability theory assumes. His decisions will not be independent of each other and will depend on numerous factors not included in our mathematical model.

I have not carried out any detailed experiments to see exactly how people's reactions to the Monty Hall problem change if all the assumptions are spelt out at the beginning – if people are told before they answer that Monty will always show them another door, that he will never show them the car, and so on – but I suspect that fewer people would get it wrong. There is, though, another source that might help.

5 The Three Prisoners problem

The Three Prisoners problem appeared in Martin Gardner's Mathematical Games column in *Scientific American* in 1959 [2]. He put it like this:

> Three men – A, B and C – were in separate cells under sentence of death when the governor decided to pardon one of them. He wrote their names on three slips of paper, shook the slips in a hat, drew out one of them and telephoned the warden, requesting that the name of the lucky man be kept secret for several days. Rumor of this reached prisoner A. When the warden made his morning rounds, A tried to persuade the warden to tell him who had been pardoned. The warden refused.
>
> 'Then tell me,' said A, 'the name of one of the others who will be executed. If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C.'
>
> 'But if you see me flip the coin,' replied the wary warden, 'you'll know that you're the one pardoned. And if you see that I don't flip a coin, you'll know it's either you or the person I don't name.'
>
> 'Then don't tell me now,' said A. 'Tell me tomorrow morning.'
>
> The warden, who knew nothing about probability theory, thought it over that night and decided that if he followed the procedure suggested by A, it would give A no help whatever in estimating his survival chances. So next morning he told A that B was going to be executed.
>
> After the warden left, A smiled to himself at the warden's stupidity. There were now only two equally probable elements in what mathematicians like to call the 'sample space' of the problem. Either C would be pardoned or himself, so by all the laws of conditional probability, his chances of survival had gone up from $1/3$ to $1/2$.
>
> ...
>
> Did A reason correctly?

As we have already noted, this is mathematically equivalent to the Monty Hall problem; the three prisoners correspond to the three doors, being pardoned to the car, being executed to the goats and the warden to Monty Hall. We can, therefore, solve it using Bayes' theorem exactly as above and if we make the same assumptions we get the same answer. A's chances of being pardoned have not changed but C's chances have doubled.

So why, if it is essentially the same problem, did this problem not appear on the front page of the New York Times?

There are all sorts of explanations, of course. The late 1950s was a different era from the early 1990s; *Scientific American* was a different sort of magazine from *Parade*, with a different sort of readership; the problem appeared as part of a series of similar problems; or it just might not have caught the public's imagination in the same way. There are, though, more interesting potential reasons. The first is that while the problems may be the same mathematically, they are in different settings. And while the assumptions made in the Monty Hall problem (whether made explicit or not) conflict with our intuitive notions about how game shows work, it seems more reasonable that A has not learnt anything about his fate from the warden (and the focus here *is* on A's chances – which don't change – whereas the Monty Hall problem focusses on the unselected and unopened door, whose chances do change). Furthermore, Gardner spells out all the assumptions explicitly and makes sure that they are satisfied:

> **Assumption 1:** The warden will always reveal a name. This is somewhat implausible – it would seem that the warden is best off staying silent, but then there would be no problem to solve.
>
> **Assumption 2:** We are told that the warden is under instruction not to reveal who is to be pardoned. Strictly speaking, this does not rule out the warden telling A his outcome if he is to be executed – but then his chances of being pardoned are 0.
>
> **Assumption 3:** We are told that the warden knows who is to be pardoned and is under instruction not to reveal who it is.
>
> **Assumption 4:** We are told that the name of the prisoner to be pardoned is chosen by drawing a name out of a hat.
>
> **Assumption 5:** We are told that, by a rather convoluted mechanism, the warden chooses at random when he has a choice.

In other words, whereas Monty is a free agent making decisions according to his own motivations, Gardner so constrains the warden that once he has decided to cooperate with A he is, in effect, a random variable. Hence, not only are the assumptions themselves inherently more plausible in this scenario, the application of Bayes' theorem is incontestable.

Interestingly, this problem appeared in an article devoted to the ambiguities that can arise if problems are not unambiguously specified. Gardner began his discussion of it with:

> A wonderfully confusing little problem involving three prisoners and a warden, *even more difficult to state unambiguously*, is now making the rounds.[4]

And stating real problems unambiguously may involve making assumptions that are approximations, implausible or simply guesswork, so you have not necessarily solved the real problem at all.

6 To switch or not to switch

I don't mean 'Don't Switch', of course, but 'Don't Necessarily Switch' isn't as catchy a title. Whether or not you should switch depends on what assumptions you make and the standard ones made are not necessarily reasonable. Gardner and Selvin had already appreciated the importance of stating problems so that the assumptions about the probabilities are unambiguous but we seem to have forgotten that in the Monty Hall problem. Having spent much of my career trying to solve real problems using mathematics, I have found that the hardest part is not usually doing the mathematics but finding out what assumptions you should make and in problems involving chance that includes being clear what the probabilities mean as well as what they are.

So if you ever find yourself in a game that looks like the Monty Hall problem should you switch or not? Mathematics only helps if you know how to estimate the probabilities and that is much harder to do than simply making plausible assumptions. My best advice is to look Monty in the eye and see if you can work out if he is trying to con you or not, or maybe if he is genuinely trying to give you another chance. Think about how many cars he has given away so far and assess whether Monty might be trying to encourage more winners or more losers. How long is there to go before the end of the game, and is Monty trying to spin it out or bring it to a halt? When you have decided that you can do all the maths. In practice, you should switch unless you think Monty is trying to con you out of a car, because in most cases you are no worse off by switching and you may gain, but how can you know? There are ways of using probability theory to tackle problems like this that are very different from the way the subject is usually taught – but that is a topic for another article.

About the author

Clive Rix is a part-time teaching fellow at the University of Leicester but has spent most of his career in the commercial world in a range of roles, including operational research, marketing and strategic planning. Most recently he has been a freelance business advisor. The views expressed in this article are his own and do not necessarily reflect the views of the University.

Notes

1. For what follows, see `http://www.marilynvossavant.com/articles/gameshow.html` (accessed 11 November 2014).
2. Available at `http://tinyurl.com/NYTimesMH` (accessed 16 November 2014).
3. There is a prior assumption here that Monty knows which door the car is behind, which is not necessarily true. If not, there are more options to consider but Bayes' theorem shows easily enough that both unopened doors have probability 1/2, so it makes no difference whether you switch or not. To keep things simple, therefore, we will assume he does know.
4. My emphasis.

References

1 Bertrand J. (1889) *Calcul des probabilités*, Gauthier-Villars et fils, Paris, pp. 2–3.

2 Gardner, M. (1959) Problems involving questions of probability and ambiguity, *Scientific American*, vol. 201, no. 4, pp. 180–182.

3 Selvin S. (1975a) A Problem in Probability, *The American Statistician*, vol. 29, no. 1, p. 67.

4 Selvin S. (1975b) On the Monty Hall Problem, *The American Statistician*, vol. 29, no. 3, p. 134.