Editorial

One hundred years ago, Albert Einstein presented his field equations of gravitation in a plenary talk to the Prussian Academy of Sciences in Berlin. These equations form the foundations for his general theory of relativity, which models gravity in terms of space–time curvature that depends on local energy and momentum rather than as a force acting upon a mass. It formalises the relationships between the presence of matter and geometry in space and time. His equations have major implications for astrophysics, space travel and satellite navigation, including the Global Positioning System (GPS) and accurate timekeeping.

This special issue of *Mathematics Today* celebrates this success story with two articles written by experts on general relativity, who elaborate upon its origins and applications. I am sure that you will enjoy reading these papers and learning more about this fascinating subject. Leaving the details to these specialists, I now share some other news about recent applications of mathematics to tackle interesting problems in three diverse settings.

The first, somewhat surprising, application links nicely with the theme of general relativity. Newton's law of universal gravitation, which preceded this theory by more than two centuries, was based on the premise of an inverse square law. This assumes that the force of attraction between two bodies is inversely proportional to the square of the distance between them. As a result, two orbiting bodies are constrained to move along conic sections: circles, ellipses, parabolae and hyperbolae. Well, congratulations to mathematician and science writer, Alex Bellos! He has designed a new game called LOOP, which involves an elliptical pool table with four balls and one pocket. It uses some simple geometrical and trigonometrical properties relating to the foci of an ellipse and looks really fabulous. I encourage you to read all about it on the *Guardian* website [1].

The second application relates to our Cavalier King Charles Spaniels, Louis and Daisy. This summer, their veterinary surgeon prescribed some medication and dispensed 90 tablets using a pill counting tray. This simple yet clever device was shaped like an equilateral triangle with an inscribed scale, from which the vet saw that 90 tablets occupy 1 tablet short of 13 rows. Counting to 13 and 1 is much quicker and more reliable than counting to 90. Ingenious! The scale lists the first few triangular numbers $\{T_1, \ldots, T_{22}\}$ including $T_{13} = 91$. Could we use a pill counting triangle without a list of triangular numbers? Well, simple inspection reveals that the number n of tablets in m complete rows is given by $n = m(m+1)/2$, which also corresponds to the number of scalars that define an $m \times m$ symmetric matrix.

We can solve this quadratic equation to give $m = \left(\sqrt{8n+1} - 1\right)/2$. As this is unlikely to be a natural number, we require $\lfloor m \rfloor$ complete rows and $n - \lfloor m \rfloor(\lfloor m \rfloor + 1)/2$ pills in the incomplete row. Setting $n = 90$ gives 12 complete rows and 12 pills in the incomplete row as expected. Although our vet is good at arithmetic, calculating a general square root is asking a bit too much, so perhaps a list is necessary after all. The most amazing feature of this pill counting triangle is that the pills can be of any reasonable size, so long as they approximate right circular cylinders of equal dimensions. This property results from optimal circle packing in the plane and leads us to ponder whether other regular or irregular pill counting polygons might exist.

My third application is rather more sombre and of greater importance. Like me, you will have been very saddened to hear about the disappearance of flight MH370 in March 2014, while travelling from Kuala Lumpur to Beijing. At the end of July this year, it was reported that a flaperon from this aeroplane drifted onto Réunion Island in the Indian Ocean, having been transported by wind and tides to this destination D. Investigators would like to learn about where the crash occurred C, as this might help to locate and retrieve the flight recorder, main fuselage, bodies and personal effects. If the crash site were known, very skilled meteorological and oceanographic modelling could have generated

Réunion island.
© Juliengrondin | Dreamstime.com

a conditional probability distribution $p(D|C)$ for the location where floating debris might be after a specified time since the known crash date.

However, we are faced with the inverse problem of identifying the crash site given the debris location and other evidence E, which includes transmitted messages, automated pinging and search results: GPS is clearly very important here. Bayes' theorem generates the required probability distribution as $p(C|D, E) \propto p(D|C)p(C|E)$. For each feasible crash site C, it is possible to determine $p(D|C)$ by simulation as above and combine this with existing beliefs about the crash site in the form of a probability distribution $p(C|E)$ to update this knowledge in light of the new debris discovery. These probabilities are then scaled by an appropriate normalising constant to ensure that

$\mathbf{t} = (240, 260, 200, 300)$ and $\mathbf{p} = (0.3, 0.1, 0.1, 0.4)$. These probabilities deliberately sum to less than one, to allow non-zero probability that the aeroplane is elsewhere. Table 1 shows that our optimal strategy is to search $200\,\mathrm{km}^2$ of Zone 4, with associated probability $P(L) \approx 0.27$ of locating the aeroplane. The corresponding probabilities for the other sensible strategies mentioned earlier are 0.17, 0.18, 0.22 and 0.22, respectively, so the recommended strategy appears to be considerably better than these alternatives. Suppose that this search fails to locate the aircraft, that we can conduct a new search of $S = 240\,\mathrm{km}^2$ and that we have no new evidence. Table 2 shows that the best strategy now is to search the remaining $100\,\mathrm{km}^2$ of Zone 4 and $140\,\mathrm{km}^2$ of Zone 1, with associated probability $P(L) \approx 0.42$ of locating the aeroplane.

… we are faced with the inverse problem of identifying the crash site given the debris location and other evidence …

$\sum p(C|D, E) = 1$ as required. Note that if the crash site were known, the other evidence mentioned above would be irrelevant. This explains the omission of E from the conditionality in the first term $p(D|C)$ on the right-hand side of this proportionality.

In fact, Bayes' theorem also plays a vital role in calculating the second term $p(C|E)$ as we now demonstrate. Immediately following the aeroplane's disappearance, investigators identified m feasible crash zones and began searching these enormous and inaccessible regions. Define $t_i > 0$ to be the area of zone i and $p_i \in [0, 1]$ to be the prior probability, or pre-search likelihood, that the aeroplane crashed in zone i. Suppose that resources were available to search a total area of size S and define $s_i \in [0, t_i]$ to be the search area for zone i, such that $\sum_{i=1}^{m} s_i = S$. Four sensible strategies are to select search areas that satisfy: (i) constant for all zones ($s_i = S/m$); (ii) proportional to areas ($s_i \propto t_i$); (iii) proportional to probabilities ($s_i \propto p_i$); (iv) proportional to probability densities ($s_i \propto p_i/t_i$). Which search strategy is optimal?

Define the events L that the aeroplane is located during the search and C_i that it is in zone i. Then our optimality criterion involves selecting the s_i to maximise the probability

$$P(L) = \sum_{i=1}^{m} P(L|C_i)P(C_i) = \sum_{i=1}^{m} \left(\frac{s_i}{t_i} \right) p_i$$

of locating the aircraft during the search. The best strategy arising from this linear programming optimisation problem is none of the four suggested above. It recommends searching completely ($s_i = t_i$) zones for which the probability density p_i/t_i is greatest and not at all ($s_i = 0$) zones for which it is least. Having decided upon the best strategy, we proceed as follows. If the search locates the aeroplane, then no further search is required. Otherwise, we update the probabilities

$$p_i \leftarrow P(C_i \mid L') = \frac{P(L' \mid C_i)P(C_i)}{P(L')} = \frac{(1 - s_i/t_i)\,p_i}{1 - \sum_{i=1}^{m} (s_i/t_i)\,p_i}$$

(forgive the ugly notation), redefine the unsearched zone areas, incorporate any new evidence and search the zones with greatest probability density as above.

To illustrate these calculations in practice, suppose that we can conduct a search of $S = 200\,\mathrm{km}^2$ across four zones with

i	t_i	p_i	p_i/t_i	s_i	$(s_i/t_i)\,p_i$
1	240	0.3	0.00125	0	0.00
2	260	0.1	0.00038	0	0.00
3	200	0.1	0.00050	0	0.00
4	300	0.4	0.00133	200	0.27
Total	1,000	0.9	0.00347	200	0.27

Table 1: First search

i	t_i	p_i	p_i/t_i	s_i	$(s_i/t_i)\,p_i$
1	240	0.41	0.00170	140	0.24
2	260	0.14	0.00052	0	0.00
3	200	0.14	0.00068	0	0.00
4	100	0.18	0.00182	100	0.18
Total	800	0.86	0.00473	240	0.42

Table 2: Second search

We can calculate further tables and strategies until a successful conclusion is reached, though accessibility of the zones should also be taken into account. Moreover, these principles and methods apply more generally to search and rescue problems, including lost sailors and hikers, victims of crime and hidden treasure. Terrible that tales of missing aircraft are, they do serve to stress the importance of GPS and hence of Einstein's general theory of relativity. Where would we be without them?

David F. Percy CMath CSci FIMA

REFERENCES

1 Bellos, A. (2015) Oh my baize! New cue sport LOOP turns pool on its head, *The Guardian*, available at www.theguardian.com/science/alexs-adventures-in-numberland/2015/jul/16/loop-new-cue-sport-pool-ellipse-elliptical