A Rigorous Rationale for Covariance Loading in Adaptive Antennas

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Abstract

Front-end injection of virtual Gaussian noise into steered adaptive arrays is shown to reverse the impairment caused by the presence of a strong desired signal. The extra noise reduces effective input SNR, enabling a robust adaptive solution which, when used to re filter the clean array signals, offers improved output SNR and faster convergence rate. The concept is comparable to the known technique of loading the sample covariance matrix diagonal values in a least squares solution and becomes mathematically equivalent at the infinite sample window asymptote. This paper demonstrates that, in general, front end noise loading can restore adaptive array performance to a level approaching the ideal least squares optimum when performance has been compromised by a combination of a short sample window, significant target strength and sensor errors.

1. Introduction

Adaptive arrays reduce interference from unwanted sources by steering spatial nulls toward them. The directions of individual noise sources being unknown, such null-steering operations proceed indirectly by optimising a least squares criterion related to output signal to noise ratio (SNR), generally using a gradient search algorithm or a Sample Matrix Inversion (SMI) computation. Typical radio applications are in communications receivers, especially in non-cooperation scenarios, while there are acoustic applications in speech enhancement and passive sonar though these are further complicated by having inherently wide-band signal sources with non-uniform spectra. The SMI method computes an array weighting vector $W = R_{XX}^{-1}V$ where R_{XX} is some estimate of the data spatial covariance, V is a pre-set blind steering or constraint vector, usually a conventional array pointing vector for the target signal, and the adaptive output scalar waveform is $y_t = W^H X_t$. Commercial communication systems use internal pilot signals to estimate V and the problems involved are rather different from the ones described here.

The statistical performance of the SMI method was explored in Reed et. al. some decades ago and, in theory at least, the method promises Least Squares (LS) optimality, fast adaptation and adequate robustness but their results apply only under a zero target-signal assumption, a caveat sometimes overlooked. The impaired adaptation speed found under target-present conditions was subsequently quantified by Miller, see Ch. 5 of Monzingo et.al., while Boroson 1980 exhaustively examined the compounding effects of random sensor errors, strong target signals and finite windows which, taken together, can prove rather catastrophic: adaptive arrays do not have a good record on robustness under these conditions. This paper will address one solution to these issues such that the zero-signal performance can be approximated when the target-signal SNR is high.

The first effect of a significant target-signal presence in the steered SMI algorithm is a reduction in convergence rate. Van Trees 2001, Monzingo et. al., observe that the sample window size needs to increase roughly in proportion to the desired signal's output SNR to maintain performance. A second effect is that when the steering vector V is mismatched from the target's actual spatial signature S there is significant suppression of a strong target signal. Such partial signal cancellation occurs equally in SMI and gradient search algorithms and was noted in Compton et.al. 1982 in relation to digital communications applications within which SNR's are typically high. Steering mismatch arises from numerous independent causes including sensor gain and phase mismatches, environmental sensor contamination, array mutual couplings, external medium anomalies like multipaths, housing diffraction, and finally there are internal electrical problems such as preamplifier cross couplings in microcircuits and local oscillator injection errors. Some errors are invariant and can be calibrated out while others are frequency, source-direction or time dependent. The literature around the related subject of conventional phased-array beamforming is voluminous though, in this application, errors merely produce a tolerable loss of array gain and an increase in sidelobe levels whereas in an adaptive array the same errors may limit output SNR to very low values, insufficient for digital reception and demodulation. As yet, no clear idea of what an optimum structure to cope with these problems has emerged, though signal-absent adaptive performance does at least give an upper bound on the level of performance to aim for.

There exist alternative spatial source separation algorithms, not based directly on least squares, which offer high output Signal to Noise Ratios for equivalent scenarios of strong signals in the presence of interference and purport to resolve these difficulties. Precision plane wave signal modelling algorithms similar to MUSIC perform at near Cramér Rao bounds for interference reduction (Monzingo et. al., Hudson et.al. 2009), but the difficulty of spatial signature modelling error is not eliminated and may even be exacerbated. Higher order statistics (HOS) and Independent Component Analysis (ICA) methods achieve remarkable linear separation of statistically independent sources, such as speech, while hardly needing any steering, and Cardoso et. al. noted that their convergence can be faster than the equivalent SMI array. However the signal sources are necessarily non Gaussian and preferably have amplitude distributions with excess kurtosis. This is not always a realistic assumption and, in particular, digital communications sources normally have sub Gaussian (platykurtic) waveforms with zero amplitude sidelobes. Resolving multiple sources with unknown individual kurtoses via ICA would cause some difficulty (Hudson 2014) so the traditional adaptive array solution may be preferred for such situations.

2. Improvement by Reducing SNR

High target-signal SNR being the proximate cause of the problems at hand in steered adaptive arrays, any means to reduce it becomes a candidate for performance enhancement. For example, it is generally known that adaptation for a fluctuating desired signal, eg. speech or pulsed radar can be switched so as to be active only during quiescent periods thereby reducing average input SNR. Minimising SNR by front end linear filtering, i.e. signal nulling, will however be found to be fundamentally futile. An alternative but counter-intuitive idea presented here is to reduce SNR by injecting an extra component of synthetic Gaussian noise into the array front end, fig. 1. Such action would, of itself, initially reduce output SNR but extracting the adaptive weight vector and using it to re-filter the original clean array signals sidesteps this problem and provides a practical robust solution within which the deleterious effects of a strong target signal can be controlled.

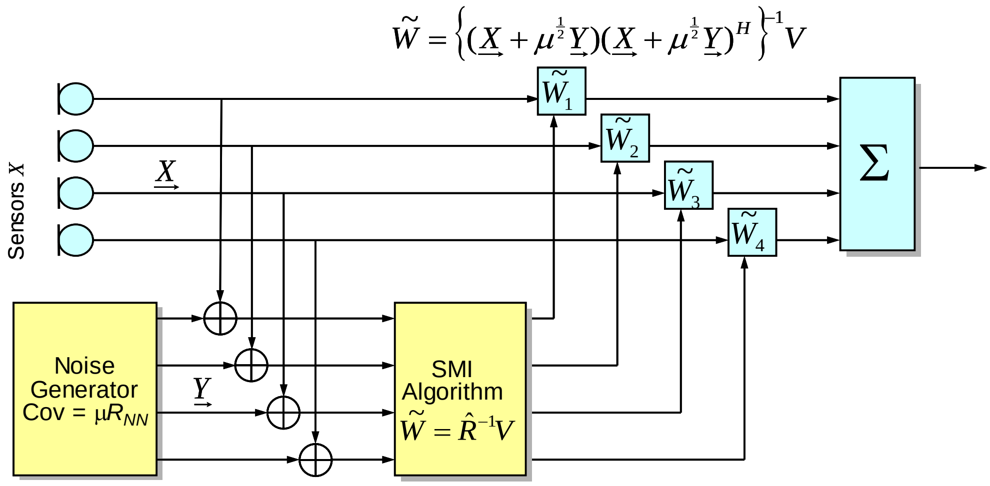

Fig 1: Virtual injection of extra noise into front end of SMI algorithm.

Ideally, the injected noise Y has covariance R_{NN} associated with noise and interference only, in which case the asymptotic optimal solution is unchanged apart from the desired SNR reduction but at simplest we can make do with spatially uncorrelated noise, $Cov(Y) = \mu \mathrm{I}$, and although this may introduce some compromise in the level of interference cancellation this, as will be shown, may not be too severe given the strong signal. In the asymptotic infinite sample window case noise injection is equivalent to diagonal covariance loading in SMI, a previously known though heuristic concept discussed by Hudson 1982, Ganz 1990, Monzingo et. al. and Van Trees 2001. Such loading is known as regularization in the wider least squares context and acts to eliminate redundant degrees of freedom (DoF) in multivariate least squares problems, suppressing smaller eigencomponents in the sample covariance which are not associated with the solution. The present paper compares the numerical diagonal loading concept with the statistical loading produced by noise injection in the presence of steering error and finite sampling windows. The latter becomes mathematically feasible by modifying the working in Boroson 1980.

3. Expressions for Array Gain

The binomial probability density for array gain (SNR improvement) with zero target-signal and error-free steering derived in the original Reed et. al. SMI solution, $W = \hat{R}_{XX}^{-1} V$, where $\hat{R}_{XX} = K^{-1} \sum_{t=0}^{K-1} X_t X_t^H$ is a covariance estimate of the zero-mean signal X, is:

$$f_1(\rho_1) = \frac{1}{B(N-1, K+2-N)} \rho_1^{K-N+1} (1-\rho_1)^{N-2} \ , \quad 0 \le \rho_1 \le 1 \tag{1}$$

where ρ_1 is the ratio of the finite sample array output SNR to its asymptotic ($K \to \infty$) SNR, N is the number of array elements, K is the number of data samples and $B(N,L) = (N-1)!\,(K-1)!/(N+K-1)!$. It is well known that for this distribution even if we use only a minimal sampling window, i.e. K at least $\ge 2N - 3$, the array will have a mean gain lying within 3 dB of the asymptotic $K \to \infty$ gain under any interference conditions, a result which promises very fast convergence and good adaptive SNR.

In the presence of an appreciable target-signal SNR however, eqn (1) no longer models the gain distribution properly. Miller (of Monzingo et. al.), noting that the PDF is entirely independent of R_{NN}, the noise covariance matrix composition, reformulated the analysis by lumping in the target signal with the interference covariance, eventually finding that if there is a continuous Gaussian target present during covariance estimation, the array gain variate ρ_1 converts to a new second variate ρ_4 (using Boroson's variables) via the transformation $\rho_4 = \dfrac{\rho_1}{1 + Q_S(1-\rho_1)}$ where $Q_S = \pi_S S^H R_{NN}^{-1} S$ (equivalent to Boroson's T_o) is the asymptotic LS output SNR for the target signal, π_S = desired signal power and S its spatial signature. The PDF of ρ_4 now relates to that of ρ_1 via a standard change of variable:

$$f_4(\rho_4) = \frac{f_1(\rho_4)}{|d\rho_4 / d\rho_1|} = \frac{1+Q_S}{(1+Q_S \rho_4)^2} f_1\!\left(\frac{(1+Q_S)\rho_4}{1+Q_S \rho_4} \right), \qquad 0 \le \rho_4 \le 1 \tag{2}$$

If there is zero desired signal and $Q_S = 0$, then $\rho_4 = \rho_1$; the gain remains unchanged but for high SNR's, $Q_S >> 1$, ρ_4 is substantially reduced whenever $\rho_1 < 1$ indicating performance loss and/or slower convergence. In Monzingo et.al. and Van Trees it is observed that the number of samples required to approach optimality increases roughly in the ratio Q_S and, since Q_S can have values up to 100 or more in communications systems, convergence rate is clearly compromised. Of course, as $K \to \infty$, both (1) and (2) will converge to the same LS optimum result: a strong signal is not a problem for very long sample windows.

Next, to address the steering mismatch problem, we revisit Boroson. Define the SMI-based Gain to Interference Ratio variable $GIR(S,P,V) = \dfrac{|S^H P^{-1} V|^2}{V^H P^{-1} R_{NN} P^{-1} V}$ for target spatial signature S, steering vector V and covariance estimate P. P can assume several different identities: asymptotic noise and interference covariance R_{NN}, asymptotic noise and interference plus target-signal covariance R_{XX}, and their finite-window sample versions $\hat{R}_{NN}$, $\hat{R}_{XX}$. A third gain measure $\rho_2 = \dfrac{GIR(S, \hat{R}_{NN}, V)}{GIR(S, R_{NN}, V)}$ is defined as the finite-window SMI gain relative to the infinite-window gain, in both cases in the presence of erroneous steering V but without target-signal data corruption. The PDF of ρ_2 is found to be

$$f_2(\rho_2) = \frac{d^{-(2K-N+1)}}{B(N-1, K+2-N)} \sum_{\ell=0}^{K+1-N} \frac{\binom{K+1-N}{\ell}^2}{\binom{N-2+\ell}{\ell}} (d-1)^{\ell} (d-\rho_2)^{N-2+\ell} \rho_2^{K+1-N-\ell} \ , \quad 0 \le \rho_2 \le d \tag{3}$$

where $d = \dfrac{GIR(S, R_{NN}, S)}{GIR(S, R_{NN}, V)} \ge 1$ is Boroson's asymptotic steering divergence parameter. Note that for ideal steering, $S = V$, in which case $d = 1$, $f_2(\rho_2) \to f_1(\rho_1)$, and eqn (3) reverts to the original Reed et. al. signal-free result of eqn (1).

The formidable expression (3) is difficult to compute for large K because of the presence of combinations but some success has been had by replacing them with Stirling-type approximations of the form $\binom{N}{k} \approx \frac{2^{N+1}}{\sqrt{2\pi N}} \exp\left[-2.45N\left(\frac{k}{N}-\frac{1}{2}\right)^2\right]$.

Having obtained distributions for the three variates ρ_4, ρ_2 and ρ_1, the final task is to find the PDF of the variate

$$\rho_5 = \frac{GIR(S,\hat{R}_{XX},V)}{GIR(S,R_{NN},V)} = \frac{\left|S^H \hat{R}_{XX}^{-1} V\right|^2}{V^H \hat{R}_{XX}^{-1} R_{NN} \hat{R}_{XX}^{-1} V} \Bigg/ \frac{\left|S^H R_{NN}^{-1} V\right|^2}{V^H R_{NN}^{-1} V} \quad (4)$$

which is the ratio of output SNR estimated using a target-corrupted sample covariance normalized to the output SNR using the infinite-window target-free covariance, in both cases with steering error. In the latter, steering error causes little target suppression beyond the conventional pattern shape because the target is absent. An expressions for ρ_5 is not immediately forthcoming so instead Boroson proceeds indirectly by devising the related but more tractable variate

$$\rho_6 = \frac{\left|S^H \hat{R}_{XX}^{-1} V\right|^2}{V^H \hat{R}_{XX}^{-1} R_{XX} \hat{R}_{XX}^{-1} V} \Bigg/ \frac{\left|S^H R_{XX}^{-1} V\right|^2}{V^H R_{XX}^{-1} V} \quad (5)$$

ρ_6 can be understood as the ratio of “signal” to ”noise+signal” output normalized by its asymptotic large sample value and, by inspection, it has the same probability distribution as ρ_2 (by substituting R_{XX} for R_{NN}). Via a chain of statistical transformations the desired variate ρ_5 is finally related to ρ_6 via the equation

$$\rho_5 = \frac{\rho_6}{(1+Q_S)\left[1+Q_S\left(1-\frac{1}{d}\right)\right] - \frac{Q_S}{d}\rho_6} \quad (6)$$

Applying a change of variable we find that ρ_5 has the PDF:

$$f_5(\rho_5) = \frac{(1+Q_S)\left[1+Q_S\left(1-\frac{1}{d}\right)\right]}{\left(1+\frac{Q_S}{d}\rho_5\right)^2} f_2\left(\frac{\rho_5(1+Q_S)\left[1+Q_S\left(1-\frac{1}{d}\right)\right]}{1+\frac{Q_S}{d}\rho_5}\right) \quad (7)$$

on the interval $0 \le \rho_5 \le \frac{d}{1+Q_S(1+Q_S)\left(1-\frac{1}{d}\right)}$. Substitution of $f_2(.)$ from eqn (3) completes the derivation and gives the final result. Note that PDF (7) is statistically consistent: as $K\to\infty$ then $f_2\to\delta(1)$, $\rho_6\to 1$ and $\rho_5 \to 1\Big/\left[1+Q_S(2+Q_S)\left(1-\frac{1}{d}\right)\right]$. For a target signal in simple uncorrelated white noise and absent discrete interferences, steering divergence d is related to the direction cosine $c = \left|S^H V\right| \Big/ \sqrt{S^H S \cdot V^H V}$ between the steering vector and the target spatial signature, by $d = c^{-2}$ and these results are then consistent with those derived in section 6.2 of Hudson 1982 where c is equivalent to the conventional beam pattern. However, Boroson’s steering divergence definition is more general such that d is also influenced, to a certain extent, by a non-white noise environment.

4. Noise Injection Performance, Finite Window

If a synthetic vector noise with covariance μR_{NN} is injected into the array front end as in fig. 1, eqn (7) continues to apply exactly but now the value of Q_S has effectively come under user control and can be made arbitrarily small so as to reduce high SNR problems. For example we have in the limit $f_5(\rho_5) \underset{\mu\to\infty}{=} f_2(\rho_5)$ and, absent interference, SMI achieves the Reed 1974 original zero-target signal optimum. Boroson's analysis is now used to predict how the output SNR will behave in finite window situations and fig 2, derived from eqn (7), shows how the desired signal gain is stabilized by noise injection in an array of N = 8 elements using a K = 96-sample finite window when there is uncorrelated noise interference only and sensor errors of around 15%. On the right is the baseline idealized zero target SMI PDF of eqn (1) which shows that the gain, relative to optimum, stays near unity. On the left is the gain when the target signal has a strength Q_S = 31 (15 dB) (i.e. an input SNR of 31/8 per element). The mean target gain has fallen to 0.035, a loss of around 14.5 dB. The third distribution, much closer to the first, is obtained when uncorrelated Gaussian noise with variance 31 (15 dB) is injected into the front end, temporarily reducing Q_S to 0 dB. The steering error loss has been mostly removed and the adaptive gain is restored almost to its optimum level with a mean value of -0.6 dB. This statistical example shows that the noise injection method of fig. 1, does improve theoretical performance and reverses impairment caused by a strong desired signal with steering error in short sample windows.

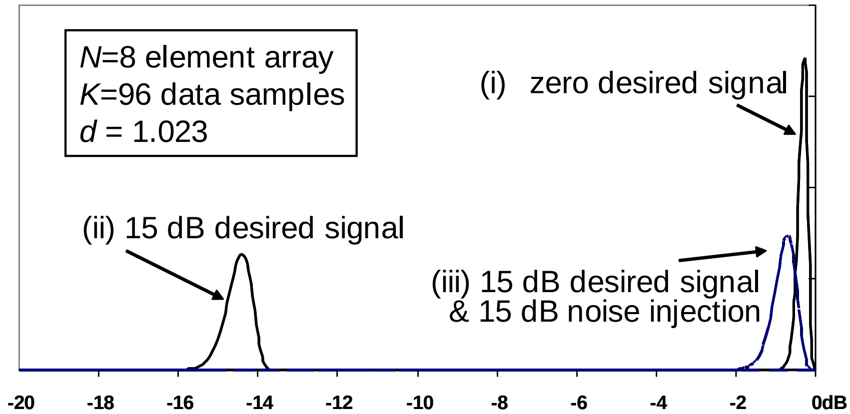

Fig. 2 Probability distributions of array gain showing (i) the ideal zero signal result
(ii) the effect of covariance corruption by the desired signal,
(iii) the restorative effect of virtual noise loading. Q_s = 31.6, N=8, K=32

Numerical diagonal loading Diagonal covariance loading gives a better performance than uncorrelated noise injection for short sample windows as shown in the next section. In the modified inverse $(\hat{R}_{XX} + \mu \mathrm{I})^{-1}$ the smaller eigencomponents of $\hat{R}_{XX}$ are suppressed, the effect being similar to a projection of the data onto a lower dimension signal subspace. It is believed that in this way numerical loading gives a bonus: subspace working effectively reduces the value of N in the PDF of eqn (1) such that there is a dual improvement in LS performance: a reduction of DoF plus a reduction in effective input SNR. However such loading is an inefficient method of subspace projection and there is a computational problem that in fast LS algorithms the covariance diagonals are typically unavailable for modification whereas noise injection is algorithm-independent. Unfortunately we do not have expressions for the gain's PDF for diagonal loading though asymptotic gain expressions are given in section 6.

5. Simulations of a Strong Target Signal, Interference, Steering Error and Finite Window

The following simulations illustrate all of the operational points discussed above. In Fig. 3 we have a line array of 8 elements and use a 96-sample window. The desired signal is set to give a conventional beam output SNR of 15 dB but there are two interferences also present of strengths 20 and 30 dB per element. With error-free steering, the hypothetical adaptive output SNR using an asymptotic infinite window is $Q_S = \pi_S S^H R_{NN}^{-1} S$ or 14.66 dB, indicated by the dashed line at the top and represents the best performance achievable. Front end injection of uncorrelated noise of variable level during the sampling window is implemented and, in fig 3, ten runs with independent data samples are shown. The sensor errors imposed are invariant complex Gaussians again with standard deviation 15% corresponding to a divergence $d \approx 1.023$. The "asymptotic diagonal loading" result is the effect of loading the known covariance diagonal, $R_{XX} \to R_{XX} + \mu \mathrm{I}$, and on this curve at the left, with zero diagonal loading, we see that straightforward SMI gives an output SNR of only 3.3 dB, a loss of 11.3 dB compared to the hypothetical optimum Q_S due to the steering error effect. With the preferred loading of μ ≈ 16 dB, a performance close to the theoretical optimum is restored.

In fig. 4, the effect of numerical loading of the sample covariance diagonals for the same 96 samples is shown: $\hat{R}_{XX} \rightarrow \hat{R}_{XX} + \mu \mathrm{I}$. This method shows a noticeable improvement over noise injection and the overall loss relative to the hypothetical optimum is becoming quite small at the optimum μ value. Some of the improvement over the noise injection method of fig. 3 is likely to have resulted from the reduction in DoF rather than the target SNR reduction.

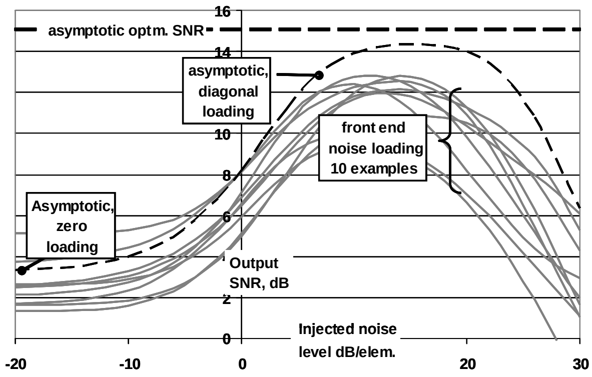

Fig 3: Probability distribution of gain in the presence of a desired signal, with noise injection. 15 dB desired signal level, 96 samples.

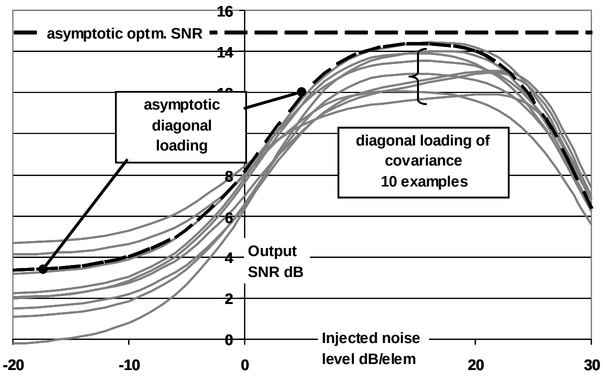

Fig. 4: Performance using diagonal loading rather than front end noise injection

To try to disentangle the effects of short windows, strong signals and steering error, fig 5 shows the probability distribution of gain ρ_4 of eqn 2 when there is no steering error, d = 1.0 but a 15 dB target signal. The average loss in these conditions is also around 3 dB so we can tentatively conclude that the residual loss in fig. 3 is due to the short window/strong signal combination rather than steering error.

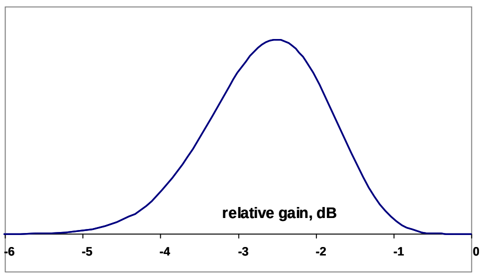

Fig 5: gain PDF for 15 dB desired signal
Without steering error.

6. Compatibility Issues

Numerical loading of the sample covariance diagonals by $\mu \mathrm{I}$ is incompatible with certain algorithms such as recursive Least Squares which do not allow direct access to the diagonals. Noise injection, by contrast, is universally compatible across all algorithms. Fig (5) shows a suggested range of applications and their compatibility with the loading methods discussed. Norm bounds on the weight vector were shown to be equivalent to diagonal loading in Hudson 1982.

	Sample Matrix Inversion	Recursive Least Squares	Steepest Descent	Steering Erros	Sequential Decorr	STAP	Excess DoF	Freq Domain
Noise Injection	✓	✓	✓	✓	✓	✓	✗	✓
Diagonal Loading	✓	✗	✓	✓	✓	✓	✓	✓
Wight Norm Bound	✓	✗	✓	✓	✓	✓	✓	✗

Fig 5: Applicability of various loading techniques for different algorithms and problem areas.

7. Signal to Noise Plus Interference Ratio Optimization by Loading

The optimum virtual noise injection level is a function of sensor errors, target SNR, sampling window duration and all interference levels. A good starting point to solve for the level is the large sample asymptote and an approximate model of the least squares solution for the case of several discrete sources in uncorrelated noise is that each source's input and output levels, including the desired source, are related by an individual power inversion equation. For an infinite window asymptote we get, from Hudson 1982, for the k^{th} source, $IO_k = \frac{c_k^2 II_k}{1 + II_k(2 + II_k)s_k^2}$ where II_k and IO_k are the input and output interference to noise ratios, the noise in this case includes injected uncorrelated noise but excludes the other sources. $c_k = \sqrt{1 - s_k^2}$, as defined in section 3, is the direction cosine between vectors S_k and V_k, aka the conventional beam response, whence s_k can then be identified as the normalized standard deviation of the sensor errors for source #*k*. The IO_k expression is equivalent to the large sample asymptote of Boroson's result of eqn. (6) *et. seq*. above except for the c_k^2 term in the numerator due to differing gain normalization. Also, Boroson's divergence *d* has a more complex definition of steering error than *c* and is applicable in multisource and non-white noise situations. Differentiation shows that the power inversion output SNR achieves a maximum value of $IO_k\big|_{MAX} = (1 - s_k)/2s_k$ at $II_k = 1/s_k$ though the real aim is to find an injected virtual noise level which compromises between maximizing the target's output SNR yet not increasing the unwanted interferences' outputs too much. As the steering error s_k value for the target signal gets larger so a higher noise injection is required and best signal to noise plus interference ratio (SNIR) will get smaller.

8 Conclusion

Statistical analysis and simulations have shown that adding vector-valued virtual noise into the front end of an adaptive array processor can reduce the deleterious effects of a combined strong desired signal and steering error such that near-optimum results are obtainable under most interference-present finite sample window conditions. The fast convergence rate of the SMI algorithm is restored and signal suppression caused by steering error is minimised. While the equivalent concept of numerical covariance matrix loading is not new, the use of virtual noise injection may have novel applications. However, in neither case has there been, hitherto, a coherent explanation of effects of loading and how to determine the optimum noise injection level. This paper has shown that the ideal level, based on the goal of maximising output SNIR, is a complex function of a number of interacting parameters, some of which, like interference levels, may be unknown at the receiver, which implies that a trial and error search procedure may be necessary to achieve optimum results.

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