

Snow Plough

Question

One day it started snowing steadily. A snow-plough started out at noon, going two miles in the first hour and one mile in the second hour.

What time did it start snowing?

Solution

Whaaaaat!!! Surely we need more information!

Yes, so we need to make some assumptions. The answer might well depend on the assumptions we make – life's like that!

Here, we'll make the simple assumption that the speed of the snow-plough is inversely proportional to the depth of snow immediately in front of it. Measuring time, t , from noon we can say that the depth of snow in front of the plough is $s(t + \tau)$, where s is the (unknown) rate at which snow is falling (in miles per hour!), t is time after noon (in hours) and τ is the length of time (also in hours) before noon for which it has been snowing. With x representing distance travelled by the snow-plough we have:

$$\frac{dx}{dt} = \frac{k}{s(t + \tau)}$$

The constant of proportionality, k , is also unknown. We can integrate this differential equation from time zero (noon) to find how far the plough travels in time, T :

$$\begin{aligned} x &= \int_0^T \frac{k}{s(t + \tau)} dt \\ &= \frac{k(\ln(T + \tau) - \ln(\tau))}{s} \\ &= \frac{k}{s} \ln\left(\frac{T}{\tau} + 1\right) \end{aligned}$$

We know from the problem statement that when $T = 1$, $x = 2$, and that when $T = 2$, $x = 3$, so:

$$2 = \frac{k}{s} \ln \left(\frac{1}{\tau} + 1 \right)$$

$$3 = \frac{k}{s} \ln \left(\frac{2}{\tau} + 1 \right)$$

By taking the ratio of these two equations we can eliminate the unknown constants s and k . So:

$$\frac{2}{3} = \frac{\ln \left(\frac{1}{\tau} + 1 \right)}{\ln \left(\frac{2}{\tau} + 1 \right)}$$

Or, after some manipulation:

$$2 \ln \left(\frac{2}{\tau} + 1 \right) = 3 \ln \left(\frac{1}{\tau} + 1 \right)$$

$$\left(\frac{2}{\tau} + 1 \right)^2 = \left(\frac{1}{\tau} + 1 \right)^3$$

$$(2 + \tau)^2 \tau = (1 + \tau)^3$$

$$\tau^2 + \tau - 1 = 0$$

The positive solution of this last equation is, of course: $\tau = \frac{\sqrt{5} - 1}{2}$, or $\tau \approx 0.618$.

Therefore, it began snowing about 0.618 hours before noon, or roughly 23 minutes past eleven.