Effective Mathematics, Effective Mathematicians

I. David Abrahams
Beyer Professor of Applied Mathematics
School of Mathematics, University of Manchester

January 4, 2010

Overview

"*Neglect of mathematics works injury to all knowledge, since one who is ignorant of it cannot know the other sciences of the things of this world. And what is worst, those who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.*" ... Roger Bacon (1214–1292) [1]. Unfortunately, the need for society (in its broadest sense) to recognize the crucial value of mathematics and its applications exists as much today as it did in the 13th century!

This article is a written account of my IMA Presidential Address, which was presented to Institute of Mathematics and its Applications (IMA) Branches throughout the UK and Ireland during my tenure of 2008/9. I have tried to remain faithful to my oral presentation, but necessarily modifications have to be made when the mode of delivery is changed; visual props and a *chatty* delivery have to be replaced by a more formal and hence lengthy setting out of material! This presentation will be divided into two parts: *Effective Mathematics* will be concerned with the obtaining of effective (or averaged) properties or descriptions of complex materials; *Effective Mathematicians* will focus on the role of the IMA in addressing the threats and opportunities facing our broad community. Unfortunately, the need to keep the article within reasonable length means that I am forced in the first half to omit all discussion I gave previously on quasicystalline alloys and composites. However, as with the oral presentation, I have tried to keep the reader's attention by skimming over much of the mathematical detail, and have included here a few additional examples that time prevented me from putting in the original address. I also include a reference list for anyone interested in further reading.

There is one factor that has meant that the second half of this article deviates quite markedly from the oral address. Much of my tenure as President was concerned with the case for a New Unified Mathematical Society (NUMS) [2] to replace both the IMA and the London Mathematical Society. My discussion on *Effective Mathematicians* was centred around this debate, but with the unsuccessful vote of the LMS now behind us my focus here will be on the issues facing our community and how the IMA, both individually and in concert with other bodies, should be tackling these.

It is worth remarking that 2010 marks the 350th anniversary of the founding of the Royal Society of London [3], the first and most distinguished learned body for science. Its founders, including Christopher Wren and Robert Boyle, described it as "*a Colledge for the Promoting of Physico-Mathematicall Experimentall Learning*". These 17th Century natural philosophers saw the great value and efficacy of mathematics in revealing the laws governing the physical world. Early Fellows of the Royal Society, such as Robert Hooke [4] and Isaac Newton, synthesised the empirical approach to science as advocated in the 13th Century by Robert Bacon and thence by Francis Bacon, with the deductive approach espoused by René Descartes. I believe that the IMA can trace its guiding principles both philosophically and genealogically to these English Restoration scientists and the new society that they created.

Effective Mathematics

"*Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.*" ... Bertrand Russell (1872–1970).

My own research area has broadly centred around the mathematics of waves, and the development and applications of this mathematics to the diffraction, scattering and propagation of waves in heterogeneous media. My work has application in a number of areas of engineering and physics, from fibre optics through non-destructive testing to Rossby waves on oceans. The first part of this article will be a rather serendipitous indulgence. It will focus on one area of current interest in wave theory that is finding ever-increasing application, and that allows investigation of some remarkable physics.

Waves are everywhere in the physical world; they are the means by which *information* propagates around a system (e.g. how a particle at one point in a fluid knows about changes elsewhere in the fluid). Waves also account for the transport of energy in a system and so play a fundamental role in almost all dynamical processes [5]. Waves occur at very small scales, where we know that all particles are described by quantum mechanical wave functions, and at the very large-scale, such as galactic density waves. The average person is familiar with a host of wave types and propagating media: electromagnetic waves from x-rays, through visible light to radio waves; waves on the surface of water; sound waves; seismological waves generated by earthquakes and so on.

The underlying mathematical descriptions for all the above wave types are remarkably similar, and so we often develop techniques to analyse specific applications and then find that these are useful in a whole host of situations. I shall try to demonstrate the ubiquity of the mathematical ideas in this article, but shall adhere to the simplest possible paradigms for ease of exposition and understanding.

Multiple scattering

When a wave of any type (e.g. acoustic, elastic, surface water, electromagnetic) is incident upon an object having physical properties distinct from the background (or host) material it will be scattered in all directions. With many obstacles, the waves scattered by one *inclusion* are incident on all its neighbours, which in turn are scattered from these onto other obstacles etc. This causes a very complicated multiply scattered wave-field, shown schematically in Figure 1.

Insert Figure 1 here, for legend see end.

In particular we are interested in characterizing global or bulk properties of waves when they are multiply scattered by thousands, millions or even many billions of objects. We shall term any material with a large number of inclusions (i.e. inhomogeneities) having properties distinct from the background, or host, as composite materials. A very important question in physics, engineering, medicine, agriculture etc. is "*How do we characterise the overall propagation properties of such composite materials?*"

One way to do this is via the approach of *homogenization.* For multiple scattering this is the mathematical limiting procedure when the incident wavelength (λ say) of waves is much larger than the characteristic inclusion size (a say) and usually also the typical inter-particle spacing (d say) [6]. Applications of homogenization are remarkably numerous, including sound scattering (noise reduction), microwave ovens, geophysics (discovering oil fields), composite material design, non-destructive testing, and biomechanics (medical imaging).

Professor Sir John Pendry, the distinguished physicist from Imperial College London, describes this homogenization process very well [7]: "*Consider light passing through a plate of glass. We know that light is an electromagnetic wave, consisting of oscillating electric and magnetic fields,* *and characterized by a wavelength, λ. Because visible light has a wavelength that is hundreds of times larger than the atoms of which the glass is composed, the atomic details lose importance in describing how the glass interacts with light. In practice, we can average over the atomic scale, conceptually replacing the otherwise inhomogeneous medium by a homogeneous material characterized by just two macroscopic electromagnetic parameters: the electric permittivity, ϵ, and the magnetic permeability, μ.*"

These two quantities define the *effective* speed of propagation of waves (and attenuation) in the glass through: $1/\sqrt{\epsilon\mu}$.

Pendry continues: "*The electromagnetic parameters ϵ and μ need not arise strictly from the response of atoms or molecules: any collection of objects whose size and spacing are much smaller than λ can be described by an effective ϵ and μ. ... From the electromagnetic point of view, we have created an artificial material ....*"

To summarise, the effective material or governing equation wraps up all the *microscale* and *mesoscale* detail, such as the type of molecules, the lattice arrangement, grain or inclusion size etc., into just a couple of constants. As an example, Figure 2 shows light passing through Calcite. Note that one sees a double image of the letters beneath it; this effect is called birefringence and is a complicated macroscopic effect of the microscale detail.

Insert Figure 2 here, for legend see end.

A simple (and enjoyable) demonstration of this property is *The Ouzo effect.* Your favourite aniseed flavoured beverage (Pernod, Pastis, Ouzo, Raki, Sambuca etc.) has the remarkable property that it spontaneously turns milky white when water is added. Why does this happen?

Ouzo contains a strongly hydrophobic oil (trans-anethole C_10H_12O), which normally would not form a stable mix with water unless subjected to a strong shear (e.g. shaking) or by use of a surfactant. However, in these drinks, the oil is dissolved in ethanol, which is strongly water-miscible. So, when water is added the oil tries to *get away* from the water, but it is dissolved in the ethanol which ensures that it stays well mixed. This balance forms a *microemulsion* (a uniform mixture of droplets in water) with drops the size of O(1 μm). These drops are of just the right size to *scatter visible light very efficiently.* Note that the mixture is remarkably stable – droplet coalescence doesn't occur and Ostwald ripening (i.e. migration of molecules to larger droplets) actually slows down with concentration.

Aside: hydrophobic materials

It is worth noting that sometimes the microstructure cannot be *averaged out* or homogenized – the small-scale details may critically influence the macroscopic behaviour. Take the case of hydrophobic surfaces, formed either naturally, such as water repellant leaves [8], or by industrial process as described by Narhe & Barthe [9]. Figure 3, reproduced from the latter paper, indicates a super-hydrophobic surface that has an '*egg box*' profile with funnel shaped spikes. Image (a) is an optical microscope image, whilst (b) is that from a scanning electron microscope; the spike width is $a = 0.5\mu$m, spike separation $b = 2\mu$m, total spike height from the bottom of the cavity is $c = 2\mu$m, and the spike height from top to the rim of the cavity $d = 0.5\mu$m. A drop of water sits high on the surface, the contact angle (see (c)) being 164^o. The contact angle is determined critically by the precise nature of the contact of the water with the surface, and this is affected by both the surface tension and the surface's microscale structure. In effect, water droplets *see* the microscale properties of the surface.

Insert Figure 3 here, for legend see end.

Industrial and biological composites

Composite materials of the particulate or fibre type, consisting of inclusions distributed throughout an otherwise homogeneous host phase, are now finding very wide application in many industrial contexts. This is due to their ability to combine several optimized material properties, for example fracture resistance and low density. Such properties cannot usually be obtained by employing simple homogeneous media. As an example, over half of the primary structure of the new Boeing 787 aircraft is made from composites; the wings in particular are constructed from carbon fibre and epoxy composites and titanium graphite laminate. The result is an aircraft that will offer 20% greater fuel efficiency, 45% increase in range, and some 11% increase in speed over the 767-series aeroplanes that it will replace.

Figures 4–7 illustrate several examples of composites commonly employed in engineering, the last used in jet engines for its excellent temperature, oxidation and creep resistance.

Insert Figure 4–7 here, for legends see end.

With their ever increasing use in critical engineering components, it is essential that such composites are able to be inspected for the onset of cracks or other irregularities. A common (and cheap) form of non-destructive testing is via the use of ultrasonic elastic waves. However, the wave propagation characteristics of perfect composites must be understood before the signature of any gross defects such as cracks can be predicted and thence detected. This same point is true for the imaging and detection in medical applications, such as tumours of the lung or other soft tissue; the propagation characteristics of x-rays through the surrounding tissue must be understood before the tumour can be accurately identified.

Insert Figures 8–9 here, for legends see end.

Another example of the medical application of waves in composite materials is as a test for osteoporosis [10]. Ultrasonic waves can be used as a quick and painless *in-vivo* means to inspect a long bone, or the heel bone (Figure 8) for this condition. Bones have a complicated form, consisting of a laminated outer layer (cortical or compact bone, see Figure 9) and a spongy (trabecular) internal region. This makes the modelling and wave propagation characteristics of the bone rather complex, but homogenization approaches are proving successful [11].

A simple one-dimensional model

Fortunately it is possible to examine very simple one-dimensional wave models in order to illustrate clearly the way that inclusions can alter propagation characteristics. We will look at transverse waves on an infinite string of mass per unit length ρ_0 and tension T. You will recall [5] that the transverse deflection u satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho_0}{T}\frac{\partial^2 u}{\partial t^2},$$

where x is the direction of the string and t is time. We can write down simple-harmonic solutions of this equation, oscillating with angular frequency ω, in complex form as

$$u(x,t) = e^{\pm ik_0 x - i\omega t},$$

in which $\pm$ indicates right/left travelling waves and

$$k_0 = \omega\sqrt{\rho_0/T} \tag{1}$$

is the wavenumber of the waves. Waves of all frequencies propagate at the same speed ($c = \sqrt{T/\rho_0}$), and so are non-dispersive.

Insert Figure 10 here, for legend see end.

Now, as shown in Figure 10, suppose we add beads (i.e. concentrated masses m) at as yet arbitrary positions x_n along the string, with an *average spacing* ℓ. Can waves of all frequencies still propagate? To investigate this, we note first that the governing equation is easily seen to be modified to

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{T}\Big(\rho_0 + \sum_n m\delta(x - x_n)\Big)\frac{\partial^2 u}{\partial t^2}, \tag{2}$$

in which $\delta(x)$ is the usual delta function. If we integrate this over $(x_n - \eta, x_n + \eta)$, $\eta \to 0$, we deduce

$$\frac{\partial u}{\partial x}\Big|_-^+(x_n, t) = \frac{m}{T}\frac{\partial^2 u}{\partial t^2}(x_n, t),$$

which yields the jump condition at each x_n. We look for a time-harmonic solution of (2) of the form $u(x,t) = U(X)\exp(-i\omega t)$, and non-dimensionalize the lengthscale on the host string wavenumber k_0 (1): $X = k_0 x$, $X_n = k_0 x_n$. We can further write the mass of a piece of the string of length ℓ, the mass ratio and the non-dimensional inter-particle spacing as, respectively,

$$m_0 = \rho_0 \ell, \qquad M = \frac{m}{m_0}, \qquad \epsilon = k_0 l. \tag{3}$$

With these substitutions, the governing equation becomes

$$\frac{\partial^2 U}{\partial X^2}(X) + U(X) = -M\epsilon \sum_n \delta(X - X_n)U(X).$$

Note that for a periodic distribution of masses, $X_n = n\epsilon$.

It is easy to simplify (4); introduce the Green function:

$$\mathcal{L}G(X) = \frac{\partial^2 G}{\partial X^2}(X;Y) + G(X;Y) = \delta(X - Y),$$

in which $G(X;Y) = \frac{1}{2i}e^{i|X-Y|}$. Thus, $\int_{-\infty}^{\infty}\{G(X)\ \mathcal{L}U(X) - U(X)\ \mathcal{L}G(X)\}\ dX$ gives

$$U(Y) = -\frac{M\epsilon}{2i}\sum_n U(X_n)e^{i|X_n - Y|}.$$

Evaluating at $Y = X_p$ yields

$$U(X_p) = U_p = \frac{Mi\epsilon}{2}\sum_{n=-\infty}^{\infty} U_n e^{i|X_n - X_p|}. \tag{4}$$

This is an exact algbraic equation for the deflection at any bead location U_p in terms of all other bead deflections. The bead positions X_n are as yet unspecified, but in the non-dimensional scaling they have an average spacing equal to ϵ.

We can use asymptotic and numerical techniques to solve (4) for random and quasi-random distributions of inclusions, but for brevity we just examine the periodic case: $X_n = n\epsilon, n \in \mathbb{Z}$. We look for a travelling wave (or periodic) solution of the form

$$U_n = \exp{(i\gamma n\epsilon)},$$

where γ is an (as yet unknown) *effective wavenumber* scaled on k_0, so that

$$e^{i\gamma p\epsilon} = \frac{Mi\epsilon}{2}\left[\sum_{n=-\infty}^{p-1} e^{i(\gamma-1)n\epsilon}e^{ip\epsilon} + e^{i\gamma p\epsilon} + \sum_{n=p+1}^{\infty} e^{i(\gamma+1)n\epsilon}e^{-ip\epsilon}\right]. \tag{5}$$

These are geometric series and so can be evaluated explicitly. (Note that the sums in (5) are divergent as written but there are standard ways, such as by adding a small dissipation term, to make them convergent.) After rearrangement we obtain the *dispersion relation* for γ:

$$\cos\gamma\epsilon = \cos\epsilon - \frac{M\epsilon}{2}\sin\epsilon, \tag{6}$$

where ϵ and M are given in (3).

In the homogenization limit, $\epsilon \to 0$, we can easily expand (6) to give

$$\gamma = \sqrt{1+M} + \frac{M^2}{24\sqrt{(1+M)}}\epsilon^2 + \frac{M^2(4+3M)(4+9M)}{5760(1+M)^{3/2}}\epsilon^4 + O(\epsilon^6).$$

There are several observations we can make on this result. First, the beads give an *added mass* to the effective wavenumber at leading order. Second, this expression is real and so allows purely propagating waves without attenuation. Third, the terms in ϵ give dispersion (different wavespeeds at different frequencies).

The dispersion relation (6) can be solved numerically for all values of the frequency/interparticle spacing. The result for the mass ratio $M = 0.1$ is shown in Figure 11. The blue line indicates the real part of γ versus ϵ whilst the mauve curve plots its imaginary part over the same range. A complex value of γ means that waves of that frequency (or ϵ) *cannot propagate*; such regions are referred to as *stop-bands*. Note how the leading order homogenized solution remains accurate up until $\epsilon \approx 2.6$, i.e. well beyond the range when ϵ can be considered small. This is not untypical of the results found for much more complex two- and three-dimensional composite models, and adds to the efficacy of homogenization techniques in general.

Insert Figure 11 here, for legend see end.

Sound waves through a bubbly liquid

Unfortunately, complicated composite problems of real interest in engineering and medicine do not allow such a simple solution procedure as that outlined above. Many alternative approaches have been developed by different communities over the years for determining the effective material properties of complex materials. These range from '*cheap and cheerful*' engineering methods to highly technical and rigorous mathematical analyses. Some approaches are designed for periodic media, some for dilute suspensions/inclusions, some use statistical approaches for random materials, and some employ energy methods to bound the result. (Readers may find helpful the discussion of the competing approaches, and the associated bibliography list, in [12].) There are many phenomena associated with waves in heterogeneous materials that cannot be discussed here; one such is *Anderson Localization*, in which random distributions always lead to attenuation of waves (complex wavenumber).

One point that must be made in regard to composites is that the effective properties *cannot be assumed to be simple averaged values* of the host and inclusion phases. Let us take another fairly simple example to show this: a distribution of air bubbles in water. The classic work by Fikioris and Waterman [13] predicts the effective wave speed c as:

$$\frac{c^2}{c_0^2} = \left(\frac{1+2\phi+2(1-\phi)\frac{\rho_1}{\rho_0}}{1-\phi+(2+\phi)\frac{\rho_1}{\rho_0}}\right)\left(1-\phi+\phi\frac{\kappa_0}{\kappa_1}\right)^{-1}, \tag{7}$$

where c_0, ρ_0, κ_0 are the sound speed, density, and bulk modulus respectively in the liquid, and ρ_1, κ_1 are the density and bulk modulus in the gas. The parameter ϕ is the gas volume fraction, which takes the value 0 when no bubbles are present, and 1 when it is all gas. The expression (7) is confirmed by many other approaches and is assumed to be fairly accurate irrespective of bubble size or distribution (periodic or random).

One may be inclined to expect that the bubbly liquid would have properties somewhere in between those of the two media. For air bubbles in water $\kappa_0 = 2.2 \times 10^9$ Pa, $\kappa_1 = 1.42 \times 10^5$ Pa, $c_0 = 1483\text{m s}^{-1}$, $\rho_0 = 10^3$ Kg m^{-3}, $\rho_1 = 1.2$Kg m^{-3}. So, as long as ϕ is not too close to zero or unity then (7) reduces to

$$\frac{c^2}{c_0^2} \approx \frac{1+2\phi}{1-\phi}\frac{\kappa_1}{\phi\kappa_0}.$$

Substituting the values of the parameters into this expression yields, rather surprisingly, an effective speed of sound for a bubbly mixture of air/water of between 32 and 50m s^{-1}, which is far lower than the speed of sound in air, $c_1 = 340\text{m s}^{-1}$! The attenuation also increases significantly; a fact that you can easily verify by tapping a wine glass of water (the '*ringing*' indicates the rate at which waves decay), then adding an effervescent tablet (e.g. an *Alka-seltzer*) to provide bubbles, and then tapping again!

Exotic composites, cloaks of invisibility and slow light

In this closing section of the first part of this article we shall briefly discuss wave propagation through several unusual materials. The first concerns *metamaterials* which are '*exotic composites*'. Quoting Pendry again [7]: "*The engineered response of metamaterials has had a dramatic impact on the physics, optics, and engineering communities, because metamaterials can offer electromagnetic properties that are difficult or impossible to achieve with conventional, naturally occurring materials. The advent of metamaterials has yielded new opportunities to realize physical phenomena that were previously only theoretical exercises.*" In particular, materials have been fabricated that have negative permittivity and permeability, and hence *negative refractive index.* Such materials have very peculiar properties, and can be strongly anisotropic. Can elastic and acoustical metamaterials be made which have similar properties to the electromagnetic composites? Presently, it seems that they can, and several have been proposed, but not yet fully fabricated and validated.

One particular application of metamaterials is to wave cloaking. This involves wrapping an object in a metamaterial such that light or other type of rays that pass through the protective layer are distorted so as to make the object '*invisible*'. We may take a simple example; if we look into a swimming pool the bottom (due to refraction) often looks much closer to the surface than it actually is. However, the apparent depth of the pool changes with observation angle (measured with respect to the vertical). Can we manufacture a material that creates a mirage of constant apparent depth at any observation angle? If so, and the material had absolute refractive index less than air, then we can effectively cloak the light away from a region, e.g. create a secret floor or compartment!

Insert Figure 12 here, for legend see end.

A more realistic and useful cloak would be that shown in Figure 12. This numerical simulation indicates that sound rays propagating horizontally are '*bent*' around the inner radius R_1 as they pass through the cloak (the area between the two concentric spheres $R_1 < r < R_2$) and exit with the same phase and direction as they would have had the cloak not been there. Thus, any object placed inside the inner sphere will be invisible to an external observer. To achieve the result shown in Figure 12 for acoustics, the cloak has to have the inhomogeneous bulk modulus

$$\kappa = \left(\frac{R_2 - R_1}{R_2}\right)^3 \left(\frac{r}{r - R_1}\right)^2,$$

where r is the radial coordinate in spherical polar coordinates, and '*anisotropic*' inhomogeneous density:

$$\rho_r = \left(\frac{R_2 - R_1}{R_2}\right)\left(\frac{r}{r - R_1}\right)^2, \qquad \rho_\phi = \rho_\theta = (R_2 - R_1)/R_2.$$

The notion that a material point can exhibit a different density when accelerated in different directions (i.e. radial and azimuthal) is clearly one that is very strange indeed; however, Graeme Milton and others (e.g. [14]) have proposed composites that could posses this remarkable property.

The final example of the effect of unusual materials on wave propagation is perhaps the most extreme. At incredibly low temperatures (a few billionths of a degree above absolute zero) and/or exceedingly high densities, Bose and Einstein predicted in 1925 that certain types of matter (bosons) could exist in a new state. A material in such a state is called a Bose-Einstein Condensate (BEC), wherein a large fraction of the bosons occupy the lowest quantum state of the external potential, and all wave functions overlap each other, thus forming a so-called '*quantum broth*'. As a result, quantum effects become apparent on a macroscopic scale. Because of the low temperatures required, it was only in 1995 that BECs were finally realised experimentally. During investigation, experimentalists discovered the amazing fact that light is slowed by an enormous amount when passing through a BEC; around 20 million-fold from 3×10^8 m s^{-1} to a mere 17m s^{-1}! Thus, a pulse of duration 3.3×10^{-6} s will shrink from 1km in length to a mere micron in length! Unfortunately, an explanation for this effect is not possible in this article, it is worth mentioning that there is presently a very active research programme within the physics community investigating '*slow light*'. One goal of this reseach is to try to create optical switches and optical memory, to use in the eventual design of optical computers.

Effective Mathematicians

"*Mathematics … is not a soil, whose fertility can be exhausted by the yield of successive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined: it is limitless as that space which it finds too narrow for its aspirations; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze.*" … James Joseph Sylvester (1814–1897).

The Institute of Mathematics and its Applications exists to provide a professional and learned society for the UK & Irish community of mathematicians and users of mathematics. What does this mean exactly and are we doing an effective job? I am not going to attempt to answer this directly in this brief discussion. In my view, the IMA aims to assist with ensuring and enhancing the quality, health and diversity of the core discipline; promulgation of mathematics to industry and commerce; assurance of quality standards of applied and applicable mathematics in traditional and emerging areas of business, industry and the education sector; and improving the understanding by users, government and general public, of the intrinsic and extrinsic value of mathematics.

But, the IMA is only worthwhile if

1. there is a job to be done (i.e. is everything '*hunky-dory*' out there?),
2. we have a relevant role to play in our community at large (i.e. we are better placed to do the jobs than other organisations),
3. the community recognises our existence and gives us a mandate to act on its behalf.

Obviously, in answer to the first and second points, I believe that there are always many issues and pressures (outlined below) that need to be addressed, and the IMA, with its broad membership, and professional and learned sides, is often best placed to do this. On the third point, some 5,000 members put their trust in us, but there are several sectors where we are weak in terms of both our input and representation as well as membership numbers.

The mathematics community and its representation

We start by addressing the second point regarding the IMA's role. There is a plethora of bodies representing and/or supporting mathematics in the UK and Ireland. A subset of these include ACME, AMET, ATM, MA, JMC, NCETM, NANAMIC, FMN, MSOR, NRICH, more maths grads, BAAS, CPAM, HoDoMS, ICMS, INI, CMS, IMA, LMS, RSS, IMS, EMS, ORS, SIAM UK, RS, RIA, KTN IM. (These acronyms are spelled out in full in Appendix B of the NUMS Consultation Document [2].) The situation abroad is also complex; there are some 130 national, cross-national and international organizations in mathematics (plus others specifically representing applied mathematics and/or mechanics.) We have a fractured representation for our broad community and this makes it difficult to speak for mathematics and to represent our community in the outside world.

As with the other learned societies, the IMA both supports and represents the academic community; however, the IMA is distinctive in the professional activities offered to mathematicians, including Chartered Mathematician designation (CMath), CMathTeach, CSci, and initial and continuing professional development. The IMA also provides careers information, offers support through its branches, and hosts a range of both academic and professional conferences. Despite this, it is true to say that many of our graduates, after taking up employment, choose to become aligned with cognate communities via the Institute of Physics (IOP), British Computer Society (BCS), Institution of Engineering and Technology (IET), Institution of Mechanical Engineering (IME), Royal Society of Chemistry (RSC), Institute of Actuaries and other professional/learned societies. This may be of necessity, so as to gain the required chartered status for example, but often it is because mathematicians in their own right lack recognition, identity and status in many sectors of industry and commerce. The Chartered Mathematician designation is helping in this regard, but we must do more to build and maintain a '*brotherhood*' of practising mathematicians.

Academic mathematicians are offered learned activities by six sister societies. Whilst these bodies work harmoniously and now have an umbrella organisation in the Council for Mathematical Sciences (CMS), it has fractured the community and makes '*speaking with a single voice*' complicated and often difficult. Mathematics promotion and engagement is similarly undertaken by a number of organizations, and sometimes suffers problems of coordination and duplication. However, the largest profusion of representative and supporting organisations is in the area of mathematics education, and especially teaching and learning in schools and FE colleges. These bodies meet under a coordinating group, the Joint Mathematical Council (JMC), and are overseen and feed input into the independent Advisory Committee on Mathematics Education (ACME). Despite JMC and ACME we still have difficulties in offering a clear steer to government, for example on the revisions to the A level syllabus.

Threats, issues and opportunities

In my opinion some of the principal issues for us to address are (i) threats to academe; (ii) forming a better brotherhood/sisterhood of mathematicians; (iii) getting our message across; (iv) training future generations of mathematicians; and (v) engaging with other discplines, industry and commerce. Let me expand on these points a little.

Education: We have a rather mixed situation with some real positives as well as chronic problems. In recent years the Government has grasped the need to improve standards in STEM subjects and to produce more STEM graduates. To support the latter, HEFCE and HEFCW have recently given £21M to form a national network; The IMA, on behalf of a consortium of mathematical organisations, will be the representative body for mathematics. Other positive points include the Smith [15] and Williams Reports [16] which have had real impact to-date, and the continued enriching and supporting work done by bodies such as the Further Mathematics Network, the National Centre for Excellence in the Teaching of Mathematics, the Mathematics Statistics and OR Network, and many others.

On the negative side there is still enormous pressure in schools on attainment and standards, league tables, changing syllabuses etc. The rapidly fracturing post-GCSE situation is giving cause for concern both in schools and universities, and how do we deal with the mixed sixth form training from the International Baccalaureate, A levels, NVQs, the new Science and Engineering Diplomas, AEAs, STEP, Pre-Us?

The statistics relating to mathematics teachers with appropriate training continues to offer depressing reading [16]: less than half of mathematics teachers at secondary level have graduated with a mathematics (type) degree, and only some 3% of the 10,000 trainee primary teachers on PGCE courses per annum have studied a STEM degree subject at university! The HE STEM programme should start to address the perception problems with students (and parents) towards the value of mathematics, and career opportunities for graduate mathematics, but this will still need serious attention from other quarters. As Sir Peter Williams states: "*The United Kingdom is still one of the few advanced nations where it is socially acceptable to profess an inability to cope with mathematics. We need to urgently reverse this trend so every pupil leaves primary school without a fear of maths.*"

Academe: The situation in the academic world has changed quite markedly since I commenced my duties within the IMA. New challenges and constraints have emerged recently, and the community must face these as well as ongoing problems such as the geographical distribution of UK & Irish mathematics departments, i.e. *desertification* of regions and polarization of small/large institutions (as outlined in the Steele Report [17]). On the latter point, the Research Assessment Exercise (RAE & henceforth REF) and research income continue to drive university policy towards viability of departments. As a rough indicator of this, the total number of submissions from departments to the three RAE units of assessment (pure, applied and statistics & OR) has decreased by over 25% between 2001 and 2008; however, the total number of FTE (category A) staff submitted rose by almost 18% between assessments. This trend is likely to be exacerbated by the recent announcement of huge cuts to funding for higher education institutions. Voluntary and perhaps enforced redundancies are a likely outcome over the next couple of years, and will place severe pressure on the mathematics HE sector to support smaller departments and maintain the numbers of undergraduates and postgraduates needed for industry, education and academe.

On top of HEFCE cut-backs, there has been substantial shift in funding within EPSRC from core expenditure on the Mathematical Sciences Programme to themed or mission programme areas, such as energy, the digital economy and next-generation healthcare. There has also been a trend in recent years, within all research councils, to deliver research support through larger more-targetted grants, and now we have to more clearly justify our activity in terms of the *impact* to the UK economy. There is a healthy debate within the IMA and the other learned societies as to the threats and opportunities of these changes; however, whether they are are potentially good or bad for mathematics as a whole, we must recognise possible problems and address them. It does seem that pure mathematicians face the hardest task to maintain sufficient funds for long-term curiosity-led research which will nurture and support the next generation researchers. The aforementioned cuts in teaching support will necessarily mean that academics will not, as in the past, be able to undertake their reserach without external grants. So, we must work collectively to ensure that, on the one hand, we grasp opportunities for new interdisciplinary research and the impact of our studies is fully recognised and exploited, whilst, on the other hand, the health of the discipline is not compromised by the changes.

I believe that the '*impact*' question is rapidly becoming a primary driver in all activities in universities. It will play a big role in assessment of departments in the REF and other areas of activity by HEFCE, and RCUK is giving strong steer to researchers to orientate their research more strongly towards this metric. This move, where impact must be measured in terms of the benefit of mathematical research to others *outside mathematics* (and ultimately to the UK economy), is coming from the highest levels of government. Taken to its logical conclusions, we could well see research councils disappearing soon, and then research funding would flow directly from government departments such as the Department for Business Innovation & Skills(BIS) and the Department of Energy and Climate Change (DECC), as well as via autonomous business-facing organisations such as the Technology Strategy Board.

We, as mathematicians of all flavours and complexions, understand that our work has impact; indeed mathematics would not be the '*language of science*' if this were not the case! But it is true that the work with most long-term impact, the work with really original ideas, is often that which starts life as purely curiosity led activity. So, how can we ensure that we have input to any debate on the precise measure of impact? Clearly, we as community need to work together to ensure that mathematics, as a fundamental branch of science, does not lose out to other near-market disciplines who can better satisfy any narrowly defined criteria. IMA is well placed, with voices from industry and the financial sector as well as academe, to make the case to government as to the real impact and value of long-term mathematical research. The larger our membership base the more effective this voice can be.

I should not conclude this section on academic issues on a totally negative note. The UK community is presently in excellent shape and, despite the lack of funding sources, is second only to the US in absolute terms on publications in top international journals. Our graduates and postgraduates are highly valued in many sectors and our total undergraduate numbers are healthy. All of this is under threat, as mentioned above, but several challenges offer enormous benefits if we are successful. A primary area of our efforts, in my opinion, should be to the '*mathematization*' of the life-sciences and in time the social-sciences too.

Industry: The great strength of the IMA is in its broad membership base. We are well placed to engage in many current debates, including the impact and benefit of mathematics to business and industry (including the biological and medical sectors). But, mathematics has not been as active as our sister subjects in engaging with the '*big issues*' such as sustainability and health, and we are still perceived as a *small* subject despite the fact that we produce around 5,000 graduates per year, which is far greater than the numbers graduating as physicists or chemists. We certainly could do more to improve links between academe and industry, and by joint effort improve the image, standing and career opportunities of practising mathematicians in all fields. The Chartered Mathematician and Chartered Scientist designations have helped in this regard, as has work by IMA and LMS on mathematics careers [18], but I would be keen for members to input their views as to what else could be done to help gain recognition for the value of mathematicians and their work, and to break down barriers within our community.

By establishing a better dialogue between teachers, researchers and industrial members we can hope to maintain and indeed strengthen the throughput of mathematicians from GCSE level through to their positions in industry. Better connections would also reduce the mis-match between the sixth-form and university levels, and ensure that the university training optimises the mathematical ability, relevance and transferable skills for graduate employment. The IMA can help achieve these goals by increasing its influence, but this is only possible by improving its relevance to members and thence increasing its membership.

Banging the drum: From the discussions above, it seems clear that mathematics has not been very good at self-promotion or in engaging with others outside the community. On the latter point, there are clearly a few individuals who make a brilliant contribution in getting the message across about the beauty, depth and efficacy of mathematics. Colleagues such as Ian Stewart, Marcus du Sautoy, Simon Singh and Chris Budd have all done much to enthuse others, but mathematicians at all levels and in all sectors can and should assist. This is now easier than ever thanks to support and encouragment from EPSRC [19], and from groups offering training and assistance (e.g. a forthcoming IMA conference on '*How to Talk Maths in Public*' [20]).

On the promotion side, our community must find a better way to speak with a single voice to Government and its departments. The overarching bodies CMS and ACME do excellent work, but are limited in their scale of operation and because of the difficulties in reaching concensus with the myriad mathematics groups. The IMA and LMS co-support a Mathematics Promotion Unit and this plays a valuable role in raising awareness of mathematical issues and providing a resource to policy makers, the media and the wider public. Again its resources are limited, and so we must find innovative ways to be more proactive. How do we coordinate an informed *position* on big issues such as renewable energy or sustainability, and how do we develop policies on topics such as research impact, the new Diploma in Science, the mathematics component of life-sciences degrees, that are in accord with our sister societies? What role should the IMA be playing in the international arena?

Concluding remarks

I hope that this brief discussion has indicated, either directly or indirectly, the role of the IMA in assisting our broad community in facing both current and future threats and opportunities. However, the ability of the IMA to be effective is directly proportional to its size and scale of activity. It is not a wealthy organisation and so can only sustain its secretariat, and their immensely valuable work, by membership fees. As mentioned earlier, there are some 5,000 graduates per year in the mathematical sciences, as opposed to fewer than 3,000 graduating each year in other science disciplines. Despite this, the Institute of Physics has around 34,000 members and the Royal Society of Chemistry attracts 44,000 members. Clearly, the fact that the mathematics and statistics community is served by three national societies, as well as several smaller ones, means that the IMA should not expect to be quite this large. Nevertheless, the combined total membership of all our societies is still well under half that of the IoP or RSC.

So, I hope you will all consider ways that the IMA could increase its membership base, and that you will contact us with ideas, criticisms, comments and especially offers of help! I believe that the branch network of the IMA is one of its great strengths. In some regions this is very successful, but in others it has become inactive due to a lack of volunteers. How could we revitalise all our branches, and how could we make better contact with school and FE teachers, our least successful membership area? Clearly, the IMA must be more relevant to those in the pre-university education sector; but it can do this, and much more, if it were larger and therefore more capable of talking to government and influencing policy on educational issues. Finally, the key to the IMA's long-term success must surely be in nurturing the future generations of mathematicians and providing them with activities and information that will assist them from '*cradle to grave*'. We have commenced a university liaison programme that now supports some twenty or more undergraduate mathematics societies [21]; have established a successful mathematics careers website; and have a very active Early Career Mathematicians Group. The last group organises two conferences a year and has a regular section in *Mathematics Today*. But this is just a start! What else could we do that would be/have been valuable to you as you are/were going through the educational process and starting out in your career as a mathematician?

It was a privilege to spend so much time with colleagues in both the IMA and LMS during the lead-up to, and roadshow in support of, the publication of the proposal for a New Unified Mathematics Society. I personally am sorry that the LMS vote went against its creation, but I sincerely wish the LMS well in its future activities, and look forward to continued close association between our two societies. I would also like to take the opportunity to thank all the honorary officers, past presidents, council members, editorial board members, and other contributors to the efforts of the IMA. They work without reward for the good of our community, and are usually influencing matters without the community's knowledge. Finally, I wish to thank the secretariat of the IMA at Catherine Richards House and especially the Executive Secretary, David Youdan. They are a fantastic team, totally dedicated to mathematics, who make a contribution far beyond that which could be expected for their numbers. They are ready for the tough fight ahead for mathematics – *are you*?

References

[1] B. Clegg, The First Scientist: A Life of Roger Bacon, Carroll & Graf, 2004.

[2] IMA and LMS, Proposal for a New Unified Mathematical Society: A Consultation Document, 2008.

[3] The Royal Society, 350th anniversary (details at http://royalsociety.org/seefurther).

[4] S. Inwood, The Man Who Knew Too Much, Pan, 2002.

[5] J. Billingham and A.C. King, Wave Motion, Cambridge University Press, 2000.

[6] W.J. Parnell and I.D. Abrahams, Dynamic homogenization in periodic fibre reinforced media. Quasi-static limit for SH waves, *Wave Motion*, **43**(6), pp. 474-498, 2006.

[7] D.R. Smith, J.B. Pendry, and M.C.K. Wiltshire, Metamaterials and Negative Refractive Index, *Science*, **305**, pp. 788–792, 2004.

[8] B. Bhushan and Y.C. Jung, Micro- and nanoscale characterization of hydrophobic and hydrophilic leaf surfaces, *Nanotechnology*, **17**, pp. 2758–2772, 2006.

[9] R.D. Narhe and D.A. Beysens, Water condensation on a super-hydrophobic spike surface, *Europhysics Letters*, **75**(1), pp. 98–104, 2006.

[10] P.H.F. Nicholson, P. Moilanen, T. Kärkkäinen, J. Timonen and S. Cheng, Guided ultrasonic waves in long bones: modelling, experiment and in vivo application, *Physiological Measurement*, **23**, pp. 755–768, 2002.

[11] W.J. Parnell and Q. Grimal, The influence of mesoscopic porosity on cortical bone anisotropy. Investigations via asymptotic homogenization, *Journal of the Royal Society Interface*, **6**, pp. 97–109, 2009.

[12] W.J. Parnell and I.D. Abrahams, Effective properties of a composite half-space; exploring the relationship between homogenization and multiple scattering theories, *Quarterly Journal of Mechanics and Applied Mathematics*, *submitted*, 2010.

[13] J.G. Fikioris and P.C. Waterman, Multiple scattering of waves. II. "Hole Corrections" in the scalar case, *Journal of Mathematical Physics*, **5**, pp. 1413–, 1964.

[14] D. Torrent and J. Sánchez-Dehesa, Anisotropic mass density by two-dimensional acoustic metamaterials, *New Journal of Physics*, **10**, p. 023004, 2008.

[15] Professor Adrian Smith, Making Mathematics Count, February 2004 (available at http://www.mathsinquiry.org.uk).

[16] Sir Peter Williams, Independent Review of Mathematics Teaching in Early Years Settings and Primary Schools, June 2008 (available at http://publications.teachernet.gov.uk).

[17] Professor Nigel Steele, Keeping HE Maths where it Counts, CMS Report, December 2007 (available at http://www.cms.ac.uk/reports/2007/steele_report.pdf).

[18] Maths Careers Website (located at http://www.mathscareers.org.uk).

[19] EPSRC, Public Engagement activity (details at http://www.epsrc.ac.uk/PublicEngagement).

[20] How to Talk Maths in Public – an IMA conference on Public Engagement, 8-9 June 2010 (details at http://www.ima.org.uk/Conferences/public_engagement).

[21] Keep update with our University Liaison Officer and his multitude of activities (details at http://www.ima.org.uk/student or for tweeters: http://twitter.com/peterrowlett).

Legends

Figure 1: Multiple scattering by inclusions.
Figure 2: Birefringence in Calcite.
Figure 3: A super-hydrophobic surface composed of a periodic array of spikes of height and spacing 2μm.
Figure 4: Silicon Carbide filaments in a Titanium alloy.
Figure 5: Cobalt fibres.
Figure 6: Metallic foam.
Figure 7: Nickel Aluminium Titanium superalloy.
Figure 8: Two bone specimens, the left from a healthy subject and the right showing some 10% bone density loss typical of osteoporosis patients.
Figure 9: Sketch of typical cortical bone structure.
Figure 10: Point masses on a string.
Figure 11: The real (blue curve) and imaginary (mauve curve) parts of the effective wavenumber γ (given by expression (6)) versus ϵ.
Figure 12: An acoustic cloak showing plane waves passing around a '*hidden*' sphere.