

The Problem of Obesity: can Mathematics help?

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Abstract

The body mass index (BMI) is widely used to quantify overweight and obesity. This article introduces a simple formula for the burden of obesity, i.e. the loss of health 'utility' and the increase in death rate. This formula could be used by individuals to assess the likely benefit of reducing weight, and as an outcome measure in clinical trials of weight reduction interventions, for example dieting, use of drugs like Orlistat, or bariatric (weight reduction) surgery.

In contrast to the heavy subject matter, I try to maintain a light tone!

1 Introduction

Of course, mathematics cannot directly help individuals to lose weight, unless we believe J. M. Barrie in *Quality Street*:

ISABELLA: Please, ma'am, father wishes me to acquire algebra.

PHOEBE (with a sinking). Algebra! It—it is not a very lady-like study, Isabella.

ISABELLA: Father says, will you or won't you?

PHOEBE: And you are thin. It will make you thinner, my dear.

Excluding this unexpectedly direct effect of algebra (not alas borne out in my experience) mathematics and statistics cannot reduce obesity, but they have at least played a rôle in quantifying it, notably through what used to be called the Quetelet index. In 1832 Alphonse Quetelet [3] decided that body weight increased as the square of height, and his index is just weight in kilograms divided by the square of height in metres. It is by the way a shame that we no longer honour this famous Belgian polymath, who used his little grey cells to advantage; after Ancel Keys (1972) [4] the Quetelet index has instead been called the body mass index (BMI).

This is rather a blunt instrument, as it makes no adjustment for gender, age or type of body frame. It is well known that athletes, with their large dense muscles, have a spuriously high BMI, while older people may increase BMI through losing height. However, the BMI is easy to measure and has become very widely used by clinicians. By most definitions, a BMI of under 18.5 is underweight, 18.5 to 25 is normal, and 25-30 is overweight. Some take BMI 27.5-30 as pre-obese, and some equate the overweight and pre-obese categories. Then 30-40 is obese, 40-50 is morbidly obese, 50-60 is superobese, and above 60 is super-super obese. . . at this point, imagination seems to be running out. The health consequences of obesity are well-known, and even female sexual attractiveness has been linked to BMI [8]; it peaks at around 21 kg / m².

The mathematical aim in this article is to find a simple functional form for the loss of *utility* and indeed of years of life or QALYs, quality-adjusted life-years, resulting from not being the optimal weight for one's height. Such a formula could maybe help individuals to assess the possible benefit of reducing weight, by means ranging from dieting to bariatric (weight loss) surgery, and could be used as an outcome in clinical trials of weight loss.

But first, what is meant by the utility of health? This is measured on a scale from 0 to 1, or as a percentage, where 100% corresponds to perfect health, and 0% health means that one would be indifferent between life and death. Utilities can indeed go negative if one would rather be dead; my grandmother in her last years used to say repeatedly that she wished she was 'in her box'. Utility could be measured just by asking people to give a number, although in practice more structured methods are used.

Figure 1 shows health utility as a function of BMI from Kortt *et al* (2005) [5]. Other results, usually more coarsely grouped by BMI, are similar [1].

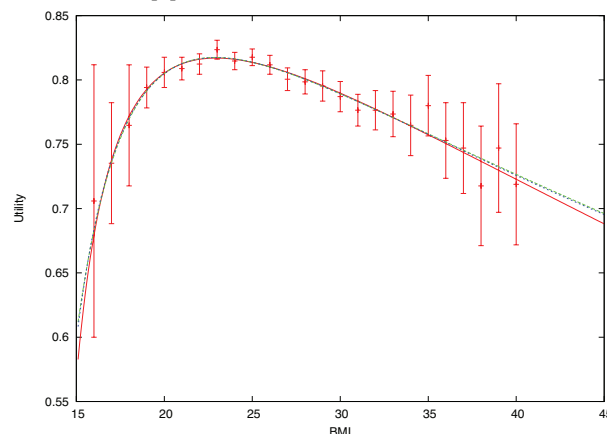


Figure 1: Utility of health from Kortt *et al* (2005) with fitted curves. Equation (2) is the solid (red) line, (1) the dashed (blue) line, and (5) the dot-dashed (green) line; these two overlap.

Health utility u is a maximum at about a BMI of $w_1 = 23 \text{ kg / m}^2$. A simple quadratic $u = c\{1 - k(w - w_1)^2\}$, where $0 < c < 1$, $k > 0$ and w is BMI, fits badly, but working on a logarithmic scale for BMI, the function $u = c\{1 - k \ln^2(w / w_1)\}$ fits much better. There is still an asymmetry in that being underweight lowers utility much faster than being overweight. The squared term was multiplied by various functions to remedy this, the most satisfactory giving the formula

$$u = c\{1 - k \exp(\alpha / w) \ln^2(w / w_1)\}, \quad (1)$$

where $\alpha > 0$. This is of course very much an *a posteriori* or descriptive (curve fitting) model, rather than a theoretical or *a priori* one, but it fitted the data. However, besides the formula's messiness, a problem arises on exploring the derivatives; we want the utility to decline monotonically with BMI, whereas if α is large enough it could temporarily increase again at high BMI.

In fact, the function $u = c\{1 - k \ln^2(w / w_1)\}$ is already asymmetric. It is not asymmetric enough, but it can be tweaked to rectify this. It turned out to fit well when changed to

$$u = c\left\{1 - k \ln^2\left(\frac{w - w_0}{w_1 - w_0}\right)\right\}, \quad (2)$$

where $w_0 \approx 13 < w_1$. An attractive feature of this model is that it has a simple physiological interpretation. What matters for

well-being (utility) is really how far the BMI is above a threshold value w_0 below which the body cannot support life. Indeed, the lowest BMIs recorded are near w_0 . The simple formula (2) fits utilities and death rates as well as (1), as seen in Figure 1. It is mathematically very tractable, e.g. the integral of utility over a range is needed for fitting utilities grouped broadly by BMI, and we have simply

$$\int_{w_0}^w \ln^2\left(\frac{x-w_0}{w_1-w_0}\right)dx = (w-w_0)\left\{\left(\ln\left(\frac{w-w_0}{w_1-w_0}\right)-1\right)^2+1\right\}.$$

A maximum-likelihood fit to data gave the parameter values in the table. The chi-squared of the fit was $\chi^2[21] = 139$, i.e. an excellent fit. The parameter $c < 1$ because even at the optimum BMI, individuals do not have perfect health.

Table 1: Parameter estimates for the final model of health utility.

Parameter	Symbol	Estimate	95% confidence interval
Low BMI	w_0	13.0	(9.1, 14.5)
Optimum BMI	w_1	22.9	(22.3, 23.6)
Maximum utility	c	0.817	(0.814, 0.820)
Multiplicative constant	k	0.1143	(0.055, 0.173)

It is fascinating to look at the lives of the most obese; what is your health utility when you are so fat that you can't even turn over in bed unaided, let alone get out of it? We can then check that the utility function (2) behaves appropriately at the largest BMIs recorded; utility should decrease monotonically, and attain a plausible value for the most obese. Figure 2 shows the extrapolation of this function to the highest BMI known, arguably about 250 kg / m².

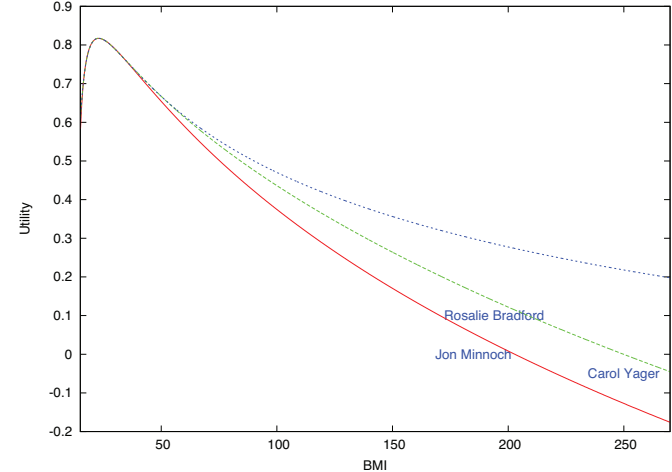


Figure 2: The fitted utility functions extrapolated to the highest known BMI. The lines are: upper (5), middle (1), lower (2).

It is interesting that the extrapolated utility is not negative at BMI 200, which agrees with biographical data on the heaviest people.

Some like Jon Brower Minnoch, who had peak BMI 186 and died at age 42, were in and out of hospital several times, suffering from cardiac and respiratory failure. However, the same year as he was hospitalised, he married, and later fathered two children. Carol Yager, who died at 34, peak BMI maybe as high as 250

kg / m², also had very ill health but had a succession of boy-friends, and Rosalie Bradford (peak BMI 194 kg / m², died aged 63) was married and had a son when her BMI was around 80 kg / m².

The lives of these very obese people show a mixture of poor health, with hospital trips requiring teams of firefighters for transport, and florid personal lives; they are celebrities. They arrive for their weddings on flatbed trucks, to the cheers of the crowd and the delight of the media.

Celebrity status no doubt improves their health utility; you can appear in *The Guinness Book of Records*, either for having enormous weight, or for losing some of it—this is actually easy to do as much of the weight is just water, caused by oedema. Clearly, despite ill health these people are not in general so miserable that they would rather be dead, and so their extrapolated utilities look reasonable. A linear decrease in utility would already have given a utility of zero by a BMI of around 150 kg / m².

Taking health utility at the optimum BMI as 1, the expression

$$k \ln^2\left(\frac{w-w_0}{w_1-w_0}\right) \tag{3}$$

with the parameter values in the table, gives the loss in utility from being a suboptimum weight. One can interpret this as the number of QALYs lost per year of life.

The rate of change of utility with BMI is

$$du/dw = -k \left\{ \frac{2 \ln(w-w_0/w_1-w_0)}{w-w_0} \right\}, \tag{4}$$

and this is always negative for $w > w_1$. Armed with equation 4, we can look at the incentive to lose weight. For example, $-du/dw$ is the gain in utility from losing 1 kg / m². The loss of utility in Figure 1 looks linear, but in fact the slope first increases and then decreases. Figure 3 shows more clearly that the rate of utility loss peaks near a BMI of 40 kg / m², and then decreases. In fact, setting $d^2u/dw^2 = 0$ gives the maximum incentive value as $w = ew_1 - (e-1)w_0$.

Nowadays fortunately we can check or even do such algebra using the excellent Wolfram Alpha ‘computational knowledge engine’, which would have been a life-saver to poor Miss Phoebe.

Hence near the onset of morbid obesity the perceived benefit from reducing weight starts to diminish, along maybe with the

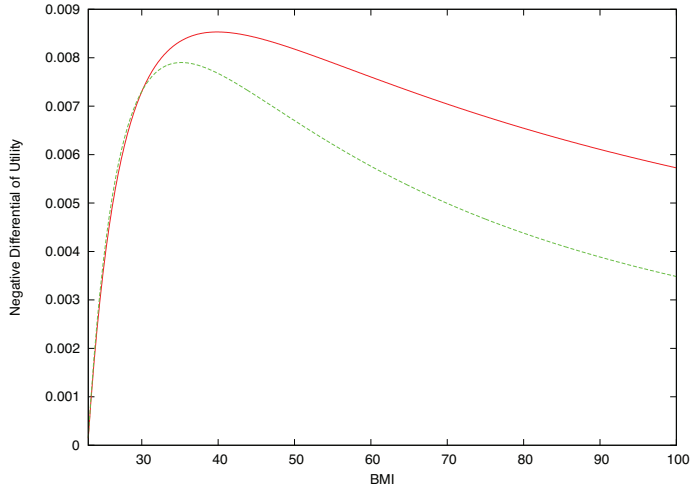


Figure 3: The derivative of the utility $du = d(-w)$ showing the declining benefit of a small weight reduction after the onset of morbid obesity. The dotted line is from formula (5).

incentive to do so. (Figure 3). This is debatable, because the utility gain from losing a fixed percentage of one's weight continues to increase, and losing weight is easier at higher BMI. All this raises the intriguing question of whether there is an 'obesity trap', where the perceived benefit from losing weight diminishes past a certain point.

In fact, the loss of utility is only a small part of the burden of obesity. Data for death rates, e.g. [6], show a similar curve, and the function (2) also fits these data well, albeit with a slightly smaller value of w_0 ; a combined fit to utility and death rate data gave $w_0 = 12.1$. Expression (3) can therefore be used as a measure of total QALYs lost per year due to suboptimal BMI. The constant of normalisation varies drastically with one's age, gender and smoking habit, but a meaningful measure can be obtained by dividing by the value at the onset of morbid obesity at a BMI of 40. Thus

$$B = 100 \frac{\ln^2(w - 12 / 23 - 12)}{\ln^2(40 - 12 / 23 - 12)} \approx 1.15 \ln^2((w - 12) / 11)$$

is proposed as a measure of the burden of obesity, where $B = 1$ at the onset of morbid obesity. Figure 4 shows this index.

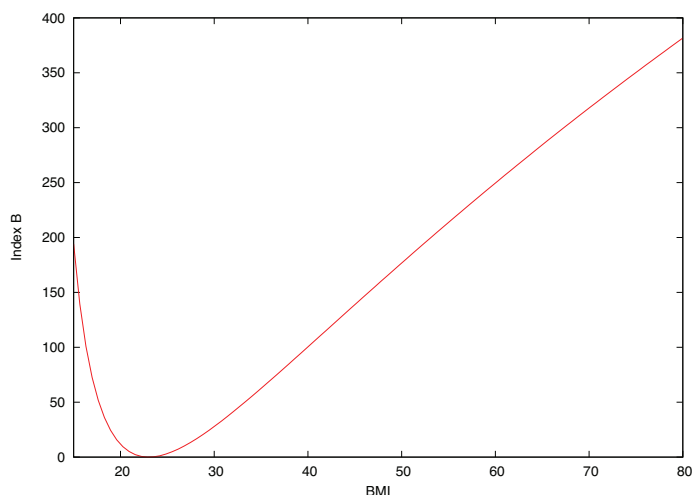


Figure 4: The 'burden of suboptimal BMI' index B as a function of BMI.

Having obtained a nice simple formula that fits the data well, one should probably leave well alone, but the urge to tinker and try to improve formulae is as irresistible to a mathematician as pizzas are to the morbidly obese. An obvious modification is to use a power of w , to obtain

$$u = c \left\{ 1 - k \ln^2 \left(\frac{w^\beta - w_0^\beta}{w_1^\beta - w_0^\beta} \right) \right\},$$

where $\beta > 0$. On fitting this to data, the estimate $\hat{\beta}$ became very small, suggesting, on taking the limit $\beta \rightarrow 0$, the formula

$$u = c \left\{ 1 - k \ln^2 \left(\frac{\ln(w / w_0)}{\ln(w / w_1)} \right) \right\}. \quad (5)$$

This fits slightly better even than (2), but loses simplicity. It shows similar qualitative behaviour, for example Figure 3 shows the 'obesity trap', where $d^2u/dw^2 = 0$, which now occurs slightly lower at a BMI of around $35 \text{ kg} / \text{m}^2$.

The best-fitting value of w_0 for utility was 11, and for combined utility and death rate data was 9. This largely removes a small problem with (2), which is how to fit individual data if someone has an 'impossible' BMI of less than w_0 .

Any discussion of BMI and weight reduction must enter a huge number of caveats. Some dispute that individuals should use the BMI as a guide to weight control at all. Older people seem to do better at an overweight BMI [7] and reverse causation exists, especially for underweight people. For example, being severely underweight can cause health problems, but wasting diseases can cause one to be underweight. This is maybe why Edwardians such as Barrie regarded thinness in women as unattractive. Researchers try to grapple with reverse causality in various ways, including the use of instrumental variables [2]. One problem I have not seen mentioned in the literature is that the health loss from obesity may not be fully reversible on losing weight; for example, diabetes once acquired will not go away again. This is a sort of hysteresis that would be difficult to study, requiring large bariatric surgery trials.

In conclusion, some simple mathematics can help with the obesity problem, although not quite in the way whimsically suggested by J. M. Barrie. The search for a functional form that would fit the data suggests that a simple quadratic logarithmic function can fit utility and increased death rates, with the logarithm of BMI distance above a threshold as the key variable. The model is easily fitted to data, so that the obesity burden of particular health risks such as stroke could be readily found. Fitting such a parametric model, and allowing the parameters k , w_0 and w_1 to be functions of demographic covariates like age and gender, would be a good way of studying the effects of obesity in detail.

REFERENCES

- 1 S. L. Dennett, K. S. Boye, and N. R. Yurgin (2008), The impact of body weight on patient utilities with or without type 2 diabetes: a review of the medical literature, *Value in Health*, **11**, 478-486.
- 2 L. Flicker, K. A. McCaul, G. J. Hankey, K. Jamrozik, W. J. Brown, J. E. Byles and O. P. Almeida (2010), Body mass index and survival in men and women aged 70 to 75, *Journal of the American Geriatric Society*, **58**, 234-241.
- 3 A. Hald (1998), A History of Mathematical Statistics from 1750 to 1930, Wiley, New York, section 26.3
- 4 A. Keys, F. Fidanza, M. J. Karvonen, N. Kimura and H. L. Taylor (1972), Indices of relative weight and obesity, *Journal of Chronic Diseases*, **25**, 329-43.
- 5 M. A. Kott and P. M. Clarke (2005), Estimating utility values for health states of overweight and obese individuals using the SF-36, *Quality of Life Research*, **14**, 2177-2185.
- 6 Prospective Studies Collaboration, Body-mass index and cause-specific mortality in 900000 adults: collaborative analyses of 57 prospective studies, *The Lancet*, **373**, 1083-1096.
- 7 G. D. Smith, J. A. G. Sterne, A. Fraser, P. Tynelius, D. A. Lawlor and F. Rasmussen (2009), The association between BMI and mortality using offspring BMI as an indicator of own BMI: large intergenerational mortality study, *British Medical Journal*, **339**, :b5043
- 8 M. J. Tovee, S. Reinhardt, J. L. Emery and P. L. Cornallisen (1998), Optimum body-mass index and maximum sexual attractiveness, *The Lancet* **352**, 548.