**Preparing for Advanced Mathematics Teaching in Early**

**Career**

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Abstract

Beginner teachers’ professional experience of post-16 teaching is varied, and often limited. This paper documents the design and implementation of a module devoted to Advanced Mathematics Teaching in Early Career (AMTEC) intended to reach 500 teachers during 2014-17. Iterative course design was informed by participant evaluation and classroom-based case study research. It was guided throughout by considering the questions ‘What (if anything) is distinctive about teaching A level?’, and ‘What is needed to teach A level well?’

The AMTEC course design took as its theoretical basis the principle of identifying ‘big ideas’ in mathematics and thus creating a minimal covering curriculum from which new teachers can start to develop deep mathematical and pedagogic knowledge required to teach A level. It extended this to selecting five pedagogic messages to be operationalised and made explicit across the course, acting as a pivot between acting as a learner and reflecting as a teacher. The paper outlines how the pedagogic messages were selected, and how they informed tutor decisions about language-use, task-choice and resource-management.

1. Context

In 2014, our institution formed a project with the Further Mathematics Support Programme (FMSP) to design a supplementary course that prepared beginner teachers to teach A level mathematics. Based on our own experiences in initial teacher education (ITE), we hypothesized that A level teaching received lower priority within training programmes for several reasons. Foremost, until recently the English standards-based system for secondary ITE mandated competence in two successive key stages, near-universally taken as ages 11-16. During school-work elements, schools and teachers often avoid exposure to Year-11 and A level examination classes where results are critical. During course-work elements, there are differences in the mathematical preparation of intending teachers around A level topics, for example in studying mechanics or statistics, and these require such careful differentiation of group sessions that directed self-study may seem a better use of time.

The project was intended to provide resources and plans for group sessions with beginner teachers in a way that could supplement or be incorporated flexibly into the coursework of existing teacher education programmes. Dissemination of these resources would raise awareness and strengthen preparation for teaching A level mathematics within the ITE community. Over three years, the resulting Advanced Mathematics Teaching in Early Career (AMTEC) course has been taken by 150 beginner teachers on UCL’s HE-led PGCE secondary mathematics route and 350 teachers from other training routes, including Teach First, School Direct Salaried, School-led ITE, Return to Teaching and Assessment–only. Participants’ levels of teaching experience range from just two weeks in ITE to two years of qualified mathematics teaching. The project aim of flexible use means that several variants of the course have been trialled. Tutors have planned and led group sessions in face-to-face and online modes (further described in our paper for session 1), over timescales of 3 days to ten weeks, while students followed up with reflection journals, access to online resources and, in some modes, peer planning and teaching. Here we focus on the reasoning that underpinned our design of a core set of ten group sessions and in particular our focus on five explicit pedagogic messages. We don’t argue that this is the only, or the best, way to design such a course, but aim to add to knowledge of pedagogy in mathematics teacher education by articulating these design considerations.

1.1. Professional Development for Early Career Teachers

Since the late 1990s, learning to teach has been conceptualized as complex and demanding intellectual work that occurs throughout the professional career span, with a progression from pre-service preparation through early-career induction to developing expertise (Cochran-Smith & Villegas, 2015). This conception of a continuum of teacher preparation underpinned our vision that the AMTEC course would be appropriate for both pre-service and early career stages, and move between learning (or re-learning) mathematical topics – the usual focus of pre-service subject knowledge courses – and discussing teaching repertoires, i.e. rehearsing when, where, how and why to use particular classroom approaches (Feiman-Nemser, 2001).

Stoll, Harris and Handscomb’s (2012) review identified nine features of “great” teacher professional development that has strong and consistent impact on learners. Among these, they note the continued importance of connecting expertise from external providers into work-based learning, focused on school needs. They argue that teacher development should be viewed as professional *learning*: teachers benefit from being aware of and understanding their own learning, exploring what motivates, influences and hinders them, in order to better understand student learning. They also recommend collaborative learning, because discussion across and within workplace contexts allows teachers to challenge each other’s thinking, and imagine changing their practice. Reviewing mathematics-specific courses, Back et al. (2009) suggest that such teacher learning is usually conceived as “doing mathematics and thinking about connections within it” (p3), and they recommend directting more explicit attention to student understanding and connections to teaching.

Although these reviews give helpful frameworks and examples of characteristics of professional development, they stop short of specifying design principles for planning a course such as AMTEC. Burkhardt (2009) characterizes design as having technical, tactical and strategic aspects, broadly moving from specifics to system-wide concerns. We found ourselves mainly interested in the tactical design, which Burkhardt describes as focusing on overall internal structure, including: selecting specific learning and performance goals; specifying sequences and cross-connections within the materials; and balancing linear coherence with diverse multiple connections. In the next section we outline the influences and decisions we made while planning our course.

**2. Design in Mathematics Teacher Development**

Course design was an iterative process, and throughout we considered the balance between the following three aspects, starting from curriculum aims and then iteratively refining these and how we achieved them by considering audience and pragmatics:

* Curriculum: what teachers would do, what mathematical and pedagogic messages would be emphasised by us and how;
* Audience: what most early-career teachers would find reasonable, challenging and motivating;
* Pragmatics: what worked in the time slots and modes chosen.

2.1. Planning from Big Ideas in Mathematics

Having committed to flexible delivery, we chose a modular approach based on 75-minute sessions, whose sequencing could easily be varied. We thus lost some facility to exploit chronological progressions, either in structuring the mathematics (for example mimicking an A level teaching sequence) or in articulating pedagogy. Instead we chose to pay attention to over-arching themes within which we could select mathematical content. Here we were influenced by our longstanding appreciation of the work of Geoff Faux (Faux, 1998) who argued for the pedagogic power of emphasising and returning to “big ideas in mathematics”, for example “equivalence” and “a lot for a little”. This had been taken further by Steve Lerman and Bernard Murphy when they used big ideas such as “proving” and “infinity” in planning a Teaching Advanced Mathematics course for experienced teachers who were new to A level. A European collaboration that shared this curriculum approach in ITE found, however, that early career teachers struggled later to identify instances of these big ideas within the school curriculum (Kuntze et al., 2011). Hence we sought a very close match between our themes and what are recognised as broad issues within the school context, partly to motivate teachers during the course and more importantly to help teachers make the connections between AMTEC sessions and teaching experiences.

The three over-arching themes chosen were: Making sense of algebraic expressions, Calculus and graphs, and Modelling with mathematics. The theme *Making sense of algebraic expressions* recognises that, in the growing abstraction of advanced mathematics, we treat algebraic expressions, along with graphs, as the objects of mathematics, and that their fluent manipulation becomes important as a means of communication and general reasoning. Our teacher audiences recognise that schools are concerned to enhance the algebraic fluency and sense-making of incoming A level students. *Calculus and graphs* was included as a theme because it is the central cluster of topics new to students at A level. Through the course iterations, we have heard back from teachers that their own knowledge of calculus is predominantly algebraic and rule-based, and that they want examples of how studying rates of change may be useful, how rates can be represented graphically or how they can be described in words. We consider this as new knowledge required for teaching, extending the teachers’ existing knowledge developed for solving calculus problems. Finally, *Modelling with mathematics* was identified in order to emphasise the utility of mathematics, and is an aspect that has been given new prominence as an objective for the 2017 A level curricula. We also considered that it was important for early career teachers to explore problems which permit multiple interpretations, such as occur in mechanics or statistics, when they are not under pressure to appear authoritative before students. Table 1 below shows how mathematical topics were allocated to these themes, and to our pedagogic messages.

Table 1 Mapping themes, content and key pedagogic messages

|  |  |  |
| --- | --- | --- |
| Theme | Mathematical focus | Key Pedagogic messages |
| P | LN | E | C | T |
| Making sense of algebraic expressions | Polynomials | X | x | x | x |  |
| Binomial Theorem |  | x | x | x |  |
| Representing Trig. Functions | X |  | x | x |  |
| Calculus and graphs | Language of Graphs & Change |  | x | x | x | x |
| Introducing Differentiation | X | x |  | x | x |
| Visualising Integration |  | x |  | x | x |
| Modelling with Mathematics | Cooling Pizzas |  |  | x | x | x |
| Newton’s Laws | X | x | x |  |  |
| Data & Distributions | X |  | x | x |  |
| No theme | Assessment |  | x | x | x |  |

2.2. Emphasising pedagogic messages in Mathematics Education

AMTEC sessions are based on mathematical activities that provide some challenge to teachers. We believe however that directing those activities – in the sense of stage directing, selecting and organising words, images, materials, time and space – matters in the mathematics classroom and so should matter in school-focused PD. We chose activities whose use with teachers could be as close as possible to the way we would use them with A level students. This introduced an element of pretence into sessions as we asked teachers to allow us to behave as if they were year 12 students. However, importantly, it allowed detailed discussion of our own teacher-ly actions that they had experienced as a group of learners.

Adler (2015, 144) has described dual attentions in professional development: "in mathematics teacher education, both ‘mathematics’ and ‘teaching’ are objects of learning. At different times each will be privileged or backgrounded”. This summarises our main concern in designing the structure of the course: how to promote teachers’ shifts of attention between learning mathematics and reflecting pedagogy. Of course, we are aware that course participants make their own decisions about what to attend to. However our view of learning as a sociocultural activity meant that we wanted to signal the value of both these “objects of learning” and, further, considered it our responsibility as tutors to indicate when the group should change its emphasis.

Initially we considered a broad scope of pedagogic reflections that we intended to encourage, and hoped would be raised by teachers themselves. This included comments on: details of how teachers had thought, felt and learnt during a task, comparisons with year 12 students’ thinking, feeling and learning; details of how we had directed activities and unpicking why those choices were made, differences in how they/we would orchestrate the same activities with students; comparisons with teaching younger students or GCSE content. These all remain valuable foci for discussion but in reflecting on early experience of running sessions we started to articulate to ourselves what were the significant pedagogic details that we wanted teachers to notice in order to discuss their effects and the reasoning behind their use. In doing so we considered what were approaches to teaching that we thought had a particular (though not unique) significance at A level and were sufficiently general in relevance that we could return to them over several sessions. We believed teachers were often highly cognitively engaged in making sense of their own, and others’, mathematical learning within their activities, and that the shift to thinking about teaching should feel as immediate by focusing on how a few messages had been operationalised locally across teaching contexts. Lastly we decided to make these approaches explicit as “key pedagogic messages” (KPM) that are privileged in the course. In practice, this means that for each session we have selected which KPM are most relevant (as shown in Table 1), and decided how to instantiate them in the delivery of the session. This helps us know what to prioritise when we formatively adapt sessions. At the end of each session, we show the most relevant KPM and ask teachers to spend 15 minutes identifying how they were operationalised. We ask them to discuss why those principles are important, what pedagogic decisions we have made and how/why one might decide differently in other contexts. More recently we have shown teachers the KPM early in each session so that they can feel more informed as they think, during the session, about both mathematics and teaching.

3. Key Pedagogic Messages

In this section we provide the rationale underlying each pedagogic message and an indication of how each was operationalised in the sessions, then refined through reflection. This records our design process and also briefly justifies why we privilege each message. We are aware that selecting the KPM necessarily foregrounds our values in discussions where we want teachers to feel able to articulate their own. Nevertheless we argue that our being ready to justify our values and how they are put into practice provides a defined, public position that teachers can critique, which may be lacking when values are left implicit.

3.1. A level planning starts by understanding progression from GCSE;

This pedagogic message emphasises the need to take into account students’ prior knowledge before introducing new mathematical ideas. We note that this idea is likely to be familiar to teacher participants from their experiences of teaching mathematics at Key Stages 3 and 4. However, it seems to take on new salience in the context of teaching A level for two reasons. Firstly, it is easy for those new to teaching A level to over-estimate the maturity of students beginning year 12 both in mathematical and behavioural terms. In this sense, this message reinforces that given in AMTEC sessions on students’ behaviour and participation at A level that these students are only ‘year 11s plus a few months’, and that the transition to studying A level mathematics therefore needs careful management. In addition, there may be an assumption that A level students are more mathematically homogenous than in earlier Key Stages so that teaching focuses on a “clever core” (Matthews & Pepper, 2007). Here, understanding progression from GCSE means differentiating to take into account the range of students that participate in A level mathematics. In planning our sessions, we aim to model pedagogy that acknowledges the different mathematical backgrounds students bring to A level study and that students may be beginning their A level course with unstable or superficial understandings of GCSE mathematics. For example, our session on differentiation begins with a reminder of how to find the gradient of curve at a point by calculating the gradient of the tangent to the curve at that point.

An equally important aim underpinning this pedagogic message is the need to present a coherent mathematical narrative by making connections between GCSE understandings and A level content. For example, our session entitled ‘Polynomials and the Factor theorem’, based upon the Standards Unit resource A11 ‘Factorising Cubics’, emphasises the connections between sketching graphs of linear, quadratic and finally cubic functions in terms of finding the y- and x-intercepts (or roots) by setting x=0 and y=0 respectively. Similarly, our session entitled ‘Representing Trigonometric Functions’ begins by recapping the use of trigonometric ratios in right-angled triangles as a means to pose the question ‘how do we define trigonometric functions for angles larger than 90 degrees?’, leading to the introduction of the unit circle as a new representation. Teacher participants have commented that emphasising such connections has prompted them to re-think how they present mathematical ideas at KS3 and 4 to provide better A level preparation for their students.

3.2. Use precise language and ‘unpick’ notation

A level mathematics introduces students to a wide range of mathematical vocabulary, sometimes adding a precise meaning to everyday words and sometimes with overlapping or subtly different meanings (such as derivative/differential). Students are also expected at A level to interpret and use conventional symbolic notations, to develop more fluent algebraic manipulation and to reason about mathematical expressions. This KPM was aimed firstly at alerting teachers to the range of new vocabulary and symbols. For many teachers this was an encounter with A level material after some time, so that vocabulary and symbol use was temporarily unfamiliar and they were well placed to recognise students’ need to spend time making sense of this terminology. We exemplify ‘unpicking’ notation by referring to the binomial expansion: the many patterns that we can ask students describe in it; its hidden coefficients of 1 and indices of zero, how sigma notation can reverse the order of terms. We emphasise the power of using consistent language over multiple representations, for example, as above, reiterating “the y-intercept is the value when x is zero”. A second aim was for individuals to articulate mathematical reasoning that would usually be silent, a feature of teachers’ practice, with awareness of communicating it to others not just for themselves. In doing so, teachers explored mathematical language, developing their knowledge for teaching. This is emphasised in our calculus sessions, where we take time to discuss terms such as gradient, slope, curve and what it makes sense to say about them. One question that has led to fruitful discussion about precise language includes the graphs shown in Figure 1:

Figure 1: For which graphs is the gradient increasing?

A regular pedagogic shift in this discussion is whether similar language activities should be used with students. Our position is that warning students about ways of being imprecise is unlikely to help them attempt mathematical talk. Nevertheless we believe teachers should be aware of and use precise language. Fluent articulation takes practice and it is often difficult to express emerging thinking, so we encourage teachers should rehearse phrases that they can use in lessons to achieve consistent precise reasoning. We have phrased this as precision over, for example, rigour, because teachers may want to vary the rigour of their talk according to circumstance but should choose the language that has this effect.

3.3. Prioritise students’ engagement through and in mathematical reasoning

We aimed to make explicit to teacher participants the decision-making behind our selection and adaptation of the mathematical activities which form the main content of the AMTEC sessions. Our aim in selecting resources (as a necessary part of designing a successful course) was to find tasks that would engage our teacher-participants and inspire them to teach A level. We were also driven by a desire to impart the message that studying A level mathematics could and should be enjoyable and intellectually stimulating for students. Thus we sought to model and promote an engaging pedagogy for teaching A level mathematics. In some tension with this desire, however, we wished to model an ‘authentic’ pedagogy that recognised the time pressures on curriculum-coverage and the need for students to spend class-time in consolidating mathematical concepts and processes.

Our first attempt at articulating this pedagogic message as “prioritise student engagement” led to the realisation that we needed to specify what kind of engagement we thought was important. Further reflection and negotiation around what we meant by engaging activities led to articulating our focus as supporting mathematical reasoning. In planning sessions, we selected resources that would engage students (and our teacher-participants) *in* mathematical reasoning because we believe thinking mathematically is what makes the subject intellectually stimulating and underpins the development of conceptual and procedural understanding. We also prioritised engagement *through* mathematical reasoning because we believe the satisfaction of achieving success in reasoning mathematically is what makes mathematics enjoyable. In this way, we aimed to model and promote an engaging pedagogy for teaching A level mathematics by highlighting how the selected resources could support reasoning and how we managed them to maximise such opportunities.

Another instance of this KPM is sharing how we have adapted resources to make them more accessible or manageable, whilst retaining features we thought were important for engaging pupils through and in mathematical reasoning. For example, we share our adaptation of the NRich task entitled ‘Whose line graph is it anyway?’, that reduced the number of and simplifies the language used in describing various scientific processes to be matched with their graphical representations. In this way, we make the task more mathematically accessible and readable for students, more manageable in time and pedagogically more accessible for our teacher participants, whilst retaining the underlying rationale of using mathematical reasoning to model scientific processes. In using the Standards Unit resource A11 ‘Factorising Cubics’, a pile of paper cards is used to form plausible factorisations of a cubic expression. Here we ask the teachers whether they think that the tactile quality of selecting and arranging three factor cards was important in supporting their own (and students’) mathematical reasoning, and whether it merits the time spent producing and managing multiple sets. Finally, and in contrast, we included tasks where consolidation was the primary concern, and mathematical reasoning secondary. An example of a consolidation task was a resource from the IntegralMaths website on finding the ‘Odd One Out’ of three trigonometric expressions, such as sin30°, sin120° and sin150° and providing a suitable match.

3.4. Make connections between mathematical topics and representations.

A Level mathematics is rich in connections, both between topics and within them. As with building on prior learning, this is a KPM that is appropriate across all teaching. Analogy is an important form of disciplinary reasoning that students should meet in mathematics. Teachers can point to connections in mathematics as evidence that ideas and reasoning met in one context can solve problems in another, returning us to Faux’s big idea that mathematics offers “a lot for a little”. The connections we focus on for this KPM are those where there is not a clear progression of learning or abstraction. We exemplify connections between apparently distinct topics through three problems featuring the binomial coefficient$\left(\begin{matrix}n\\r\end{matrix}\right)$: a counting problem, counting the instances of $a^{r}b^{n-r}$ in the expansion of $(a+b)^{n}$ and a probability problem. We also show connections between representations: in this same session, we look at different ways of expanding brackets or, in the trigonometry session, teachers use both graphs and a unit circle approach to solve trigonometric equations and discuss how they are similar. Such discussions are underpinned by research that suggests teachers ununderestimate the difficulties for students of making connections between representations: “what matters is not representations but their transformation" (Duval, 2006, p. 107).

3.5. Deliberate use of tools so that students can make and share predictions

Several of our sessions model how tools can be used to support students in making and sharing predictions. In particular, the session on differentiation begins by using GeoGebra to gather empirical data on the gradient of the tangent to the curve *y* = *x*2 for a series of x-coordinates. We then predict the gradient function of *y* = *x*2 based on this data, confirming our prediction through graphing the gradient function and later seeking proof through differentiation by first principles. Similarly, GeoGebra is used in the session on integration to generate empirical data leading to the conjecture of an area function, setting the left limit to zero and varying the right limit, for the graph of y = x. In another session, teacher-participants are asked to predict the distance-time graph after watching a video from the *Graphing Stories* website (<http://graphingstories.com>). They then share their predictions, reflecting on aspects of the graph which need more careful attention, before watching the video again in half-time leading to the improvement of their original sketch-graphs.

Initially then this KPM focussed on promoting students’ engagement in making and sharing predictions through the use of tools. Subsequently, this focus shifted to emphasise *how* tools are used by the teacher (or *how* their use is structured) to engage students in making and sharing predictions. This shift was reflected in the inclusion of the word ‘deliberate’ to indicate our careful and planned use of tools to focus attention on mathematical concepts by slowing down dynamic variation and reducing visual complexity. For example, in the session on differentiation we make explicit how we slow down dynamic variation by using arrow keys, rather than the mouse, to control sliders to step along the x-axis. Stepping along the x-axis in this way allows the tutor to stop and discuss with teacher-participants what happens to the tangent and the numerical gradient of *y* = *x*2 as the x coordinate increases. Visual complexity is reduced by introducing new aspects of the GeoGebra file one at time and discussing them as or even before they are revealed. For example, we discuss how the gradient of the function at a point is calculated by working out the gradient of the tangent at that point before revealing on-screen the numerical gradient calculated by the software.

Our choice of ‘tool’ in the wording of this key pedagogic message signals our on-going negotiation of how widely we interpret this term. So far, the exemplification of deliberate tool use in our sessions has mainly centred on the use of ICT. In this sense, it might be appropriate to restrict our use of ‘tool’ to mean ICT tools only, emphasising their value in the teaching of A level mathematics. On the other hand, it might be more helpful to use ‘tools’ in a wider sense, for example to emphasise how we make deliberate use of resources or physical tools such as mini-whiteboards and equipment used in mechanics.

4. Discussion

This paper has outlined how a course designed to prepare early career teachers for teaching A level came to be structured around mathematical themes and – more innovatively – key pedagogic messages. These five messages were chosen for their significance in the future workplace in making decisions about teaching, and for their relevance to the teachers’ immediate activities of re-learning A level mathematics topics for the purpose of teaching. As such, their origins are grounded in the design process. We selected and refined our objectives for what teachers would do, and emphasised/ rejected aspects of our own input in directing those activities. The pedagogic messages both arose from this process and shaped it. At our current stage, the course has become relatively stable: we still make impromptu teaching decisions but these are motivated by helping this group of teachers to focus on the KPM as pivots between teachers acting as a learner and reflecting as a teacher.

As we bring new tutors into the team, the KPM have been useful in communicating what we consider important in sessions. This has helped us secure consistency in different times of running the course but also allowed the tutors to make minor adaptations and evaluate them against whether they support the modelling or the discussion of the KPM. One of the questions that has arisen from these wider discussions is whether the five pedagogic messages are strong enough to consider separately from the course in which they are embedded and in which their operationalisation is made explicit as an object of study. We feel that they are broad enough to support teachers’ later planning, but also note their unevenness, with one addressing specifics of technological tool use and another considering making connections as part of learning. We have also queried the distinctiveness of A level teaching. All these messages would be appropriate to bear in mind for all teaching. Nevertheless we feel it is important to explore and critique how they could be instantiated with A level content and learners, and note that some messages appear to resonate more strongly with early career teachers when presented in the A level curriculum (for example, using Geogebra) and during teachers’ learning about A level. We have found that, when early career teachers engage in some activities of A level mathematics, they attend strongly to their own mathematics so that the shift to considering teaching needs clear delineation. Our key pedagogic messages are a response to this need.

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