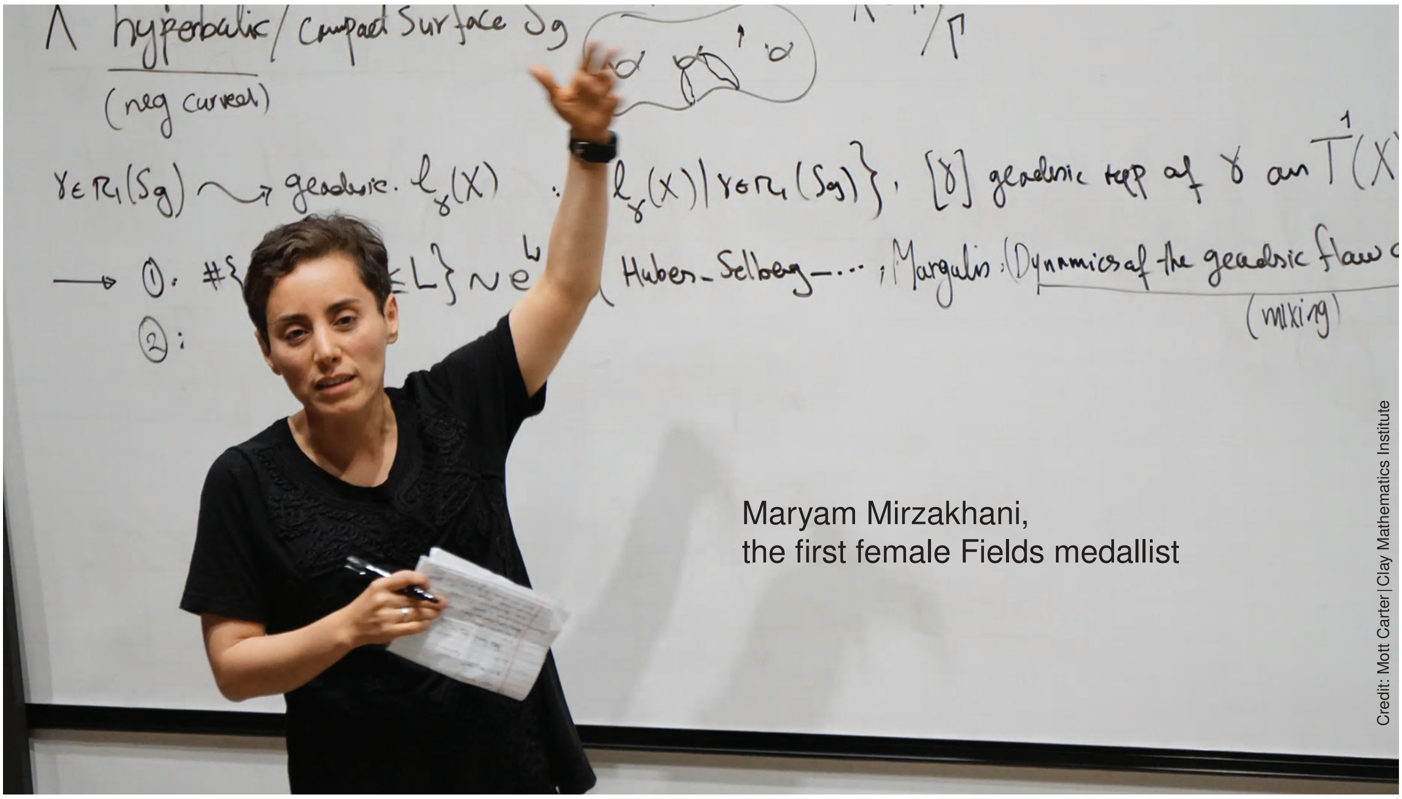
Maryam Mirzakhani and her work

Maryam Mirzakhani, the first woman to win the International Mathematical Union's premier prize the Fields medal, died aged 40 on 14 July 2017.

Born and brought up in Tehran, Maryam went to Farzanegan, a high school for girls with exceptional talents. Her outstanding mathematical ability became evident when she won gold medals in the International Mathematical Olympiads in Hong Kong (1994) and again in Canada (1995), becoming the only Iranian ever to achieve a perfect score.

Following undergraduate studies at the prestigious Sharif University in Tehran, Maryam worked for her PhD at Harvard under Fields medallist Curtis McMullen. Her brilliant 2004 doctoral thesis brought her widespread recognition and a Clay fellowship at Princeton. Rising rapidly from assistant professor to professor, in 2008 she moved with her husband Jan Vondrák, a computer scientist, to a chair at Stanford. Continuing to produce spectacular mathematics, in 2014 she won a Clay Research Award in recognition of her 'many and significant contributions to geometry and ergodic theory, in particular to the proof of an analogue of Ratner's theorem on unipotent flows for moduli of flat surfaces'.

Mirzakhani's award of the Fields medal for her 'stunning advances in the theory of Riemann surfaces and their moduli spaces' in the same year (2014) made international headlines. Awarded once every four years to mathematicians under the age of 40, Mirzakhani was the first Iranian to win this coveted honour.



Maryam Mirzakhani,
the first female Fields medallist

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So what was her work actually about? In 2002 I received a rather apologetic letter from Maryam, then a student at Harvard, together with a rough draft thesis and a request for comments. After reading only a few pages, I was transfixed. Starting with a formula discovered by Greg McShane in his 1991 Warwick PhD, she had developed some amazingly original and beautiful machinery which culminated in a completely new proof of *Witten's conjecture*, a relation between integrable systems of Hamiltonian PDEs and the geometry of certain families of 2D topological field theories. (The conjecture was first proved in 1992 by Maxim Kontsevich in his thesis, part of the work which won him the Fields medal in 1998.)

McShane's identity [1] is an intriguing formula involving the lengths of simple (that is, non-self-intersecting) closed geodesics on a hyperbolic once punctured torus $\Sigma_{1,1}$. This is a surface of genus 1 with 1 puncture locally modelled on the hyperbolic plane. Since geodesics can be continued indefinitely, the puncture has to be at infinite distance from the middle of the torus. Geodesics can escape from the body of the torus towards the puncture only by travelling out along an exponentially thinning cusp (see Figure 1). Consider an unpunctured torus $\Sigma_{1,0}$ formed by gluing together opposite sides of a parallelogram. Modelled on Euclidean space, for each rational direction on $\Sigma_{1,0}$ there is a foliation by an infinite family of 'parallel' simple closed geodesics. On $\Sigma_{1,1}$ the situation is quite different: each rational direction still defines a homotopy class γ of simple loops, but now there is a unique smooth closed geodesic of minimal length $\ell(\gamma)$ in each class. McShane's identity states that $2\Sigma_\gamma\big(1+e^{\ell(\gamma)}\big)^{-1}=1$.

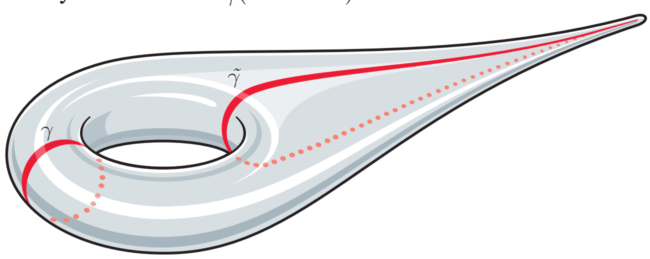

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Figure 1: A punctured torus with a simple closed loop γ and a homotopic curve $\tilde\gamma$ escaping up the cusp.

The space of all possible hyperbolic structures on $\Sigma_{1,1}$ is called its *Teichmüller space*, denoted $\mathcal{T}(\Sigma_{1,1})$. It has one complex degree of freedom – the same as the space of all complex structures on a Euclidean torus. To see this, note that any parallelogram can be placed after translation and scaling with 3 adjacent vertices at $0, 1, \tau$ where τ is a complex number with positive imaginary part. Placing punctures at the vertices, it follows from the classic *uniformisation theorem* that there is a unique hyperbolic metric on $\Sigma_{1,1}$ compatible with the natural complex structure inherited from the parallelogram. Thus $\mathcal{T}(\Sigma_{1,1})$ can be identified with the upper half plane $\mathbb{H}$. The *moduli space* $\mathcal{M}(\Sigma_{1,1})$ is obtained when one forgets the choice of *marking*, meaning a labelling of curves on the torus relative to a fixed base. For $\Sigma_{1,1}$ a marking is specified by picking base vectors $e_1 = (1, 0)$ and $e_2 = (0, \tau)$; it can be changed by sending (e_1, e_2) to $(ae_1 + be_2, ce_1 + de_2)$, for any linear map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Thus $\mathcal{M}(\Sigma_{1,1})$ is the quotient of $\mathcal{T}(\Sigma_{1,1})$ by the action $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ of $SL(2, \mathbb{Z})$ on $\mathbb{H}$. This group of marking changes is called the *mapping class group* $\mathrm{Mod}(\Sigma_{1,1})$ of $\Sigma_{1,1}$.

McShane's identity holds for every structure in $\mathcal{T}(\Sigma_{1,1})$. Maryam noted that although one needs a marking to make sense of the lengths $\ell(\gamma)$, the sum in the identity is invariant under changes of marking. Thus integrating both sides of McShane's identity would give a formula for the volume $V_{1,1}$ of $\mathcal{M}(\Sigma_{1,1})$ – a quantity, which for higher genus surfaces, is of interest in algebraic geometry and mathematical physics.

But how can you integrate terms in the sum without knowing the marking? Here is the trick. Every simple curve is the image of fixed curve γ_0 under some element in $\mathrm{Mod}(\Sigma_{1,1})$. The stabiliser of γ_0 in $\mathrm{Mod}(\Sigma_{1,1})$ is the set of Dehn twists round γ_0. (A Dehn twist (Figure 2) is the marking change resulting when the torus is cut open along γ_0, one of the cut ends is rotated by a full twist relative to the other, and then reglued.) Thus integrating $\Sigma_\gamma(1 + e^{\ell(\gamma)})^{-1}$ over $\mathcal{M}(\Sigma_{1,1})$ is equivalent to integrating $(1 + e^{\ell(\gamma_0)})^{-1}$ over the quotient of $\mathcal{T}(\Sigma_{1,1})$ by the Dehn twists round γ_0. The computation yielded $V_{1,1} = \pi^2/6$.

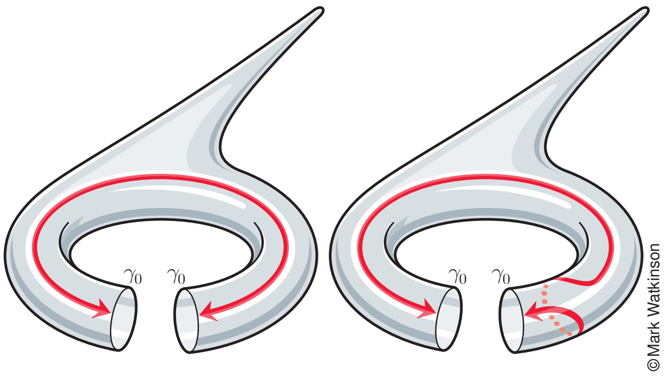

Figure 2: A Dehn twist. The punctured torus is cut open along γ, then one open end is twisted before being reglued.

Ingenious though this was, it recovered a known formula. Her next steps were truly remarkable. McShane's identity depends on a careful analysis of how the geodesics escape up the cusp. By an intricate inductive process involving cutting open surfaces and regluing, Maryam extended the identity to any surface and then proceeded to build up recursion relations for the (Weil-Petersson) volume $V_{g,n}(L_1, \ldots, L_n)$ of the moduli space of a surfaces of genus g with n geodesic boundary curves of fixed lengths $L_1, \ldots, L_n$. This involved invoking a process called *symplectic reduction* to allow for cutting, twisting and regluing as described above.

Her induction showed that $V_{g,n}(L_1, \ldots, L_n)$ is a polynomial in $L_1^2, \ldots, L_n^2$ whose coefficients are rational up to known powers of π [2]. Remarkably, the coefficients store information about intersection numbers of certain Chern classes on the moduli space. Reducing the recurrence relations involved in Witten's conjecture to hers, she arrived at a completely new proof of the conjecture [3]. By applying methods of dynamics and ergodic theory to the action of the mapping class group on $\mathcal{T}(\Sigma_{g,n})$, she also used her methods to show that the number of simple geodesics of length at most L on $\Sigma_{g,n}$ grows like $L^{6g-6+2n}$, and found formulae for the frequencies of geodesics of different topological types [4]. For example, the probability of a genus 2 surface falling apart after cutting along a simple closed loop is $1/7$.

Mirzakhani's subsequent work revolves around dynamics on various moduli spaces. Perhaps most striking is the analogue of Ratner's celebrated rigidity theorem, mentioned above in the citation for her Clay award. Ratner's theorem says that the closure of the orbit of a one parameter flow with no exponential behaviour on a homogeneous space is itself a closed subgroup. The flow in Mirzakhani's setting takes place on another moduli space, that of so-called translation surfaces, a generalisation of tori made from parallelograms. Take a polygon whose sides are identified in pairs, in such a way that paired sides are of equal length and parallel (see Figure 3, based on a figure in [6]). We also require that when the sides are glued up to make the translation surface, the angle round each vertex be an integral multiple of 2π. Since linear maps preserve paired sides, the group $SL(2, \mathbb{R})$ acts on the space of translation surfaces in a natural way.

The question is, what does the closure of an $SL(2, \mathbb{R})$ orbit look like? On the face of it, one might expect some sort of fractal attractor. Mirzakhani, together with Alex Eskin and Amir Mohammadi [5] showed that on the contrary, the orbit closure is manifold locally defined by certain special linear equations. The immensely complicated and deep proof has been described by

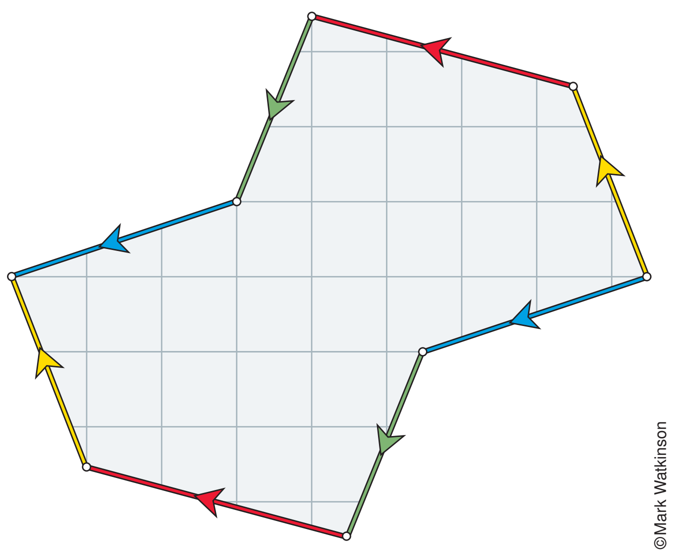

Figure 3: A polygon with paired sides glued up to make a translation surface. Strictly speaking, the grid is part of the structure.

expert Anton Zorich as a ‘titanic work’ [6]. It has already found applications, for example to problems about billiards on polygonal tables, to the illumination problem (understanding the sightlines of a security guard in a complex of mirrored rooms), and to the Ehrenfest wind tree model. For a nice introduction to this subject, see [7].

Maryam proved other ground-breaking results about dynamics on moduli spaces which we have no space for here. The last time I saw her was when, as pictured, she came to Oxford to receive her Clay Research Award in 2015. As usual, it was a joy to watch her expound her marvellous ideas. With her infectious enthusiasm, she was always keen to discuss mathematics, always optimistic about what could be done, modest and unassuming while projecting an unwavering self-confidence. She had a reputation for tackling the most difficult questions with dogged persistence. McMullen described her as having ‘ . . . a fearless ambition when it comes to mathematics – a sort of daring imagination’.

Maryam’s premature death from cancer has deprived us of an outstanding mathematician at the height of her powers. Survived by her husband and their six year old daughter Anahita, she is deeply mourned by the whole mathematical community. Her unique mathematical legacy will endure for many years to come.

Maryam Mirzakhani, mathematician, born 3 May 1977; died 14 July 2017

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References

1 McShane, G. (1998) Simple geodesics and a series constant over Teichmüller space, *Invent. Math.*, vol. 132, pp. 607–632.

2 Mirzakhani, M. (2007) Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces, *Invent. Math.*, vol. 167, pp. 179–222.

3 Mirzakhani, M. (2007) Weil-Petersson volumes and intersection theory on the moduli space of curves, *J. Amer. Math. Soc.*, vol. 20, pp. 1–23.

4 Mirzakhani, M. (2008) Growth of the number of simple closed geodesics on hyperbolic surfaces, *Ann. of Math.*, vol. 168, pp. 97–125.

5 Eskin, A., Mirzakhani, M. and Mohammadi, A. (2015) Isolation, equidistribution, and orbit closures for the $SL(2,\mathbb{R})$ action on moduli space, *Ann. of Math.*, vol. 182, pp. 673–721.

6 Zorich, A. (2015) The work of Maryam Mirzakhani, *AMS Notices*, vol. 62, pp. 1345–1349.

7 Wright, A. (2016) From rational billiards to dynamics on moduli space, *Bull. AMS.*, vol. 53, pp. 41–56.