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CONFIGURATION SPACES OF SPATIAL LINKAGES

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Abstract. Most useful linkages are spatial. Commonly, for such a mechanism there are configurations $c \in \mathcal{C}$ were two links or more intersect. We introduce a mathematical framework to deal with the neighborhood of c and define *virtual configurations* in which two links touch. Cauchy sequences are then used to approximate these configurations. As for the global structure, we show how the completion of the configuration space relates to a *virtual configuration space* were all lines meet the origin. We conclude the paper with some explicit example.

0. Introduction

A *linkage* is a collection of rigid bars, or *links*, attached to each other at their vertices, with a variety of possible *joints* (fixed, spherical, rotational, and so on). These play a central role in the field of robotics, in both its mathematical and engineering aspects: see [Fa].

Such a linkage Γ, thought of as a metric graph, can be embedded in an ambient Euclidean space $\mathbb{R}^d$ in various ways, called *configurations* of Γ. The space $\widehat{\mathcal{C}}(\Gamma)$ of all such configurations has a natural topology and differentiable structure. There are also various functions of mechanical significance defined on $\widehat{\mathcal{C}}(\Gamma)$, such as the work map $\psi : \widehat{\mathcal{C}}(\Gamma) \to \mathcal{W}$ (which associates to each configuration the position of the "end-effectors" of Γ). In this paper we consider a somewhat simplified mathematical model of the configuration space (see [BS1]), which nevertheless provides useful information on the usual physical version (cf. [SLTS]).

1. Configuration spaces

Any embedding of a linkage in a (fixed) ambient Euclidean space $\mathbb{R}^d$ is determined by the positions of its vertices, but not all embeddings of its vertices determine a legal embedding of the linkage. To make this precise, we require the following:

1.1. **Definition.** An *linkage type* is a graph $\mathcal{T}_\Gamma = (V, E)$, determined by a set V of N vertices and a set $E \subseteq V^2$ of k edges (between distinct vertices). We assume there are no isolated vertices. A specific *linkage* $\Gamma = (\mathcal{T}_\Gamma, \vec{\ell})$ of type $\mathcal{T}_\Gamma$ is determined by a *length vector* $\vec{\ell} := (\ell_1, \dots, \ell_k) \in \mathbb{R}^E_+$, specifying the length $\ell_i > 0$ of each edge (u_i, v_i) in E $(i = 1, \dots, k)$. This $\vec{\ell}$ is required to satisfy the triangle inequality where appropriate. We call an edge with a specified length a *link*, (or bar) of the linkage Γ, and the vertices of Γ are also known as *joints*. We write $\vec{\ell}^2 := (\ell_1^2, \dots, \ell_k^2) \in \mathbb{R}^E_+$ for the vector of *squared* lengths.

An *embedding* of $\mathcal{T}_\Gamma$ in the Euclidean space $\mathbb{R}^d$ is a map $\mathbf{x} : V \to \mathbb{R}^d$ such that the open intervals $(\mathbf{x}(u_i), \mathbf{x}(v_i))$ and $(\mathbf{x}(u_j), \mathbf{x}(v_j))$ in $\mathbb{R}^d$ are disjoint if the edges (u_i, v_i) and (u_j, v_j) are distinct in E, and the corresponding closed intervals $[\mathbf{x}(u_i), \mathbf{x}(v_i)]$ and $[\mathbf{x}(u_j), \mathbf{x}(v_j)]$ intersect only at the images of common vertices.

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The space of all such embeddings is denoted by $\mathrm{Emb}(\mathcal{T}_\Gamma)$; it is an open subset of $(\mathbb{R}^d)^V$.

We have a *moduli function* $\lambda_{\mathcal{T}_\Gamma} : (\mathbb{R}^d)^V \to [0, \infty)$, written $\mathbf{x} \mapsto \lambda_{\mathcal{T}_\Gamma}(\mathbf{x}) : E \to [0, \infty)$, with:

$$(\lambda_{\mathcal{T}_\Gamma}(\mathbf{x}))(u_i, v_i) := \|\mathbf{x}(u_i) - \mathbf{x}(v_i)\|^2 \qquad \text{for } (u_i, v_i) \in E .$$

We think of $\Lambda := \mathrm{Im}(\lambda_{\mathcal{T}_\Gamma})$ as the *moduli space* for $\mathcal{T}_\Gamma$.

The *immersion configuration space* of the linkage $\Gamma = (\mathcal{T}_\Gamma, \vec{\ell})$ is the subspace $\mathcal{C}^{\,\mathrm{im}}(\Gamma) := \lambda_{\mathcal{T}_\Gamma}^{-1}(\vec{\ell}^2)$ of $(\mathbb{R}^3)^V$. A point $\mathbf{x} \in \mathcal{C}^{\,\mathrm{im}}(\Gamma)$ is called an *immersed configuration* of Γ: it is determined by the condition

$$\|\mathbf{x}(u_i) - \mathbf{x}(v_i)\| = \ell_i \qquad \text{for each edge } (u_i, v_i) \text{ in } E . \tag{1.2}$$

We use the squared lengths so that $\lambda_{\mathcal{T}_\Gamma}$ is an algebraic function, and thus $\mathcal{C}^{\,\mathrm{im}}(\Gamma)$ is a real algebraic variety, which inherits its differentiable structure from $(\mathbb{R}^d)^V$.

Finally, the *embedding configuration space* of the linkage $\Gamma = (\mathcal{T}_\Gamma, \vec{\ell})$ is the subspace $\mathcal{C}(\Gamma) := \mathcal{C}^{\,\mathrm{im}}(\Gamma) \cap \mathrm{Emb}(\mathcal{T}_\Gamma)$ of $\mathrm{Emb}(\mathcal{T}_\Gamma)$. A point $\mathbf{x} \in \mathcal{C}(\Gamma)$ is called an (embedded) *configuration* of Γ. Since $\mathrm{Emb}(\mathcal{T}_\Gamma)$ is open in $(\mathbb{R}^d)^V$, $\mathcal{C}(\Gamma)$ is open in $\mathcal{C}^{\,\mathrm{im}}(\Gamma)$, so it is a semi-algebraic set. We can extend these definitions by allowing infinite length edges (lines or half-lines), modifying the moduli function and space accordingly.

2. Virtual configurations

The space of immersion configurations $\mathcal{C}^{\,\mathrm{im}}(\Gamma)$ was used in [BS2, §1] as a simplified model for the space of all possible configurations of Γ. However, this is not a very good approximation to the behaviour of linkages in 3-dimensional space. We now suggest a more realistic (though still simplified) approach, as follows:

2.1. **Definition.** Note that since $j : \mathrm{Emb}(\mathcal{T}_\Gamma) \to (\mathbb{R}^3)^V$ is an embedding into a manifold, it has a *path metric*: for any two functions $\mathbf{x}, \mathbf{x}' : V \to \mathbb{R}^d$ we let $\delta'(\mathbf{x}, \mathbf{x}')$ denote the infimum of the lengths of the rectifiable paths from $\mathbf{x}$ to $\mathbf{x}'$ in $\mathrm{Emb}(\mathcal{T}_\Gamma)$ (and $\delta'(\mathbf{x}, \mathbf{x}') := \infty$ if there is no such path). We then let $d_{\mathrm{path}}(\mathbf{x}, \mathbf{x}') := \min\{\delta'(\mathbf{x}, \mathbf{x}'), 1\}$.
This is clearly a metric, which is topologically equivalent to the Euclidean metric on $\mathrm{Emb}(\mathcal{T}_\Gamma)$ inherited from $(\mathbb{R}^3)^V$ (cf. [L, Lemma 6.2]). The same is true for the metric d_{path} restricted to the subspace $\mathcal{C}(\Gamma)$ (compare [RR]).

2.2. *Remark.* Since any continuous path $\gamma : [0, 1] \to \mathrm{Emb}(\mathcal{T}_\Gamma)$ has an ϵ-neighborhood of its image still contained in $\mathrm{Emb}(\mathcal{T}_\Gamma) \subseteq (\mathbb{R}^d)^V$, by the Stone-Weierstrass Theorem (cf. [Fr, §3.7]) we can approximate γ by a smooth (even polynomial) path $\hat{\gamma}$. Thus we may assume that all paths between configurations used to define d_{path} are in fact smooth.

2.3. **Definition.** We define the *completed space of embeddings* of $\mathcal{T}_\Gamma$ to be the completion $\widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ of $\mathrm{Emb}(\mathcal{T}_\Gamma)$ with respect to the metric d_{path} (cf. [Mu, Theorem 43.7]). The (complete) *configuration space* $\widehat{\mathcal{C}}(\Gamma)$ of a linkage Γ is similarly defined to be the completion of the embedding configuration spaces $\mathcal{C}(\Gamma)$ with respect to $d_{\mathrm{path}}|_{\mathcal{C}(\Gamma)}$. The new points in $\widehat{\mathcal{C}}(\Gamma) \setminus \mathcal{C}(\Gamma)$ will be called *virtual configurations*: they correspond to actual immersions of Γ in $\mathbb{R}^d$ in which (infinitely thin) links are allowed to touch, "remembering" on which side this happens.

Note that the moduli function $\lambda_{\mathcal{T}_\Gamma} : \mathrm{Emb}(V) \to \mathbb{R}^E$ of §1.1 extends to $\hat{\lambda}_{\mathcal{T}_\Gamma} : \widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma) \to \mathbb{R}^E$, and in fact $\widehat{\mathcal{C}}(\Gamma)$ is just the pre-image $\hat{\lambda}_{\mathcal{T}_\Gamma}(\vec{\ell}^2)$ for the appropriate vector of lengths $\vec{\ell}$.

2.4. *Remark.* In effect, in the space $\widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ we have separated the non-embedded immersions (in which one edge e_1 of $\mathcal{T}_\Gamma$ intersects another edge e_2, not at a common node) into two different virtual configurations, in one of which e_1 lies "immediately above" e_2, and in the other "immediately below" (compare [Va]).

Even though the metric d_{path} is topologically equivalent to the Euclidean metric d_2 on $\mathrm{Emb}(\mathcal{T}_\Gamma)$, its completion with respect to the latter is simply $(\mathbb{R}^d)^V$, so the corresponding completion of $\mathcal{C}(\Gamma)$ is the space of immersion configurations $\mathcal{C}^{\mathrm{im}}(\Gamma)$ of §1.1.

2.5. **Definition.** Given a linkage type $\mathcal{T}_\Gamma$ with $\mathrm{Emb}(\mathcal{T}_\Gamma) \subseteq (\mathbb{R}^3)^V$, let $P(\mathcal{T}_\Gamma)$ denote the space of paths $\gamma : [0,1] \to (\mathbb{R}^3)^V$ such that $\gamma((0,1]) \subseteq \mathrm{Emb}(\mathcal{T}_\Gamma)$ (cf. [L, Ch. 3]), and let $\mathrm{ev}_0 : P(\mathcal{T}_\Gamma) \to (\mathbb{R}^3)^V$ send $[\gamma]$ to $\gamma(0)$. Let $E(\mathcal{T}_\Gamma)$ denote the set of homotopy classes of such paths relative to $\mathbf{x} = \gamma(0)$. This is a quotient space of $P(\mathcal{T}_\Gamma)$, with $\widehat{\mathrm{ev}}_0 : E(\mathcal{T}_\Gamma) \to (\mathbb{R}^3)^V$ induced by ev_0. We shall call $E(\mathcal{T}_\Gamma)$ the *path space of embeddings* of $\mathcal{T}_\Gamma$.

Similarly, let $\mathcal{P}(\Gamma) \subseteq P(\mathcal{T}_\Gamma)$ denote the space of paths $\gamma : [0,1] \to (\mathbb{R}^3)^V$ such that $\gamma((0,1]) \subseteq \mathcal{C}(\Gamma)$, with $E(\Gamma)$ the corresponding set of relative homotopy classes (a quotient of $P(\Gamma)$). We call $E(\Gamma)$ the *path space of configurations* of Γ.

2.6. **Lemma.** *The completed space of embeddings* $\widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ *is a quotient of the path space* $E(\mathcal{T}_\Gamma)$, *and* $\widehat{\mathcal{C}}(\Gamma)$ *is a quotient of* $E(\Gamma)$.

Proof. First note that when $\gamma(0) \in \mathrm{Emb}(\mathcal{T}_\Gamma)$, the path γ is completely contained in the open subspace $\mathrm{Emb}(\mathcal{T}_\Gamma)$ of $(\mathbb{R}^3)^V$, so we may represent any homotopy $[\gamma]$ by a path contained wholly in an open ball around $\gamma(0)$ inside $\mathrm{Emb}(\mathcal{T}_\Gamma)$, and any two such paths are linearly homotopic. Thus $\widehat{\mathrm{ev}}_0$ restricted to $\mathrm{Emb}(\mathcal{T}_\Gamma))$ is a homeomorphism.

In any completion $\hat{X}$ of a metric space (X, d), the new points can be thought of as equivalence classes of Cauchy sequences in X. Since we can extract a Cauchy sequence (in the path metric) from any path γ as above, and homotopic paths have equivalent Cauchy sequences, this defines a continuous map $\phi : E(\mathcal{T}_\Gamma) \to \widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma)$.

In our case, $X = \mathrm{Emb}(\mathcal{T}_\Gamma)$ also has the structure of a manifold, and given any Cauchy sequence $(x_i)_{i=1}^\infty$ in X, choose an increasing sequence of integers n_k such that $d_{\mathrm{path}}(x_i, x_j) < 2^{-k}$ for all $i, j \geq n_k$. By definition of d_{path}, we have a path γ^k from x_{n_k} to $x_{n_{k+1}}$ of length $\leq 2^{-k}$. By concatenating these and using Remark 2.2, we obtain a smooth path γ along which all $(x_i)_{i=1}^\infty$ lies. Thus we may restrict attention to Cauchy sequences $(x_i)_{i=1}^\infty = (\gamma(t_i))_{i=1}^\infty$ lying on a smooth path γ in X. We can parametrize γ so that $\gamma((0,1]) \subseteq \mathrm{Emb}(\mathcal{T}_\Gamma)$, and let $\gamma(0) := \lim_{i\to\infty} x_i \in (\mathbb{R}^3)^V$, which exists since $(\mathbb{R}^3)^V$ is complete. Thus ϕ is surjective. □

3. Blow-up of singular configurations

As in the proof of Lemma 2.6, we may think of points in $\widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ as Cauchy sequences $(x_i)_{i=1}^\infty = (\gamma(t_i))_{i=1}^\infty$ lying on a smooth path γ in $\mathrm{Emb}(\mathcal{T}_\Gamma)$, which we can partition into equivalence classes according to the limiting tangent direction $\vec{\mathbf{v}} := \lim_{i\to\infty} \gamma'(t_i)$ of γ.

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We can use this idea to construct an approximation to $\widehat{\mathrm{Emb}}(\mathcal{T}_{\Gamma_n})$, by blowing up the singular configurations where links or joints of Γ meet (cf. [Sh, II, §4]). For our purpose the following simplified version will suffice:

3.1. **Definition.** Given an abstract linkage $\mathcal{T}_\Gamma = (V, E)$, we have an orientation for each edge. Let $\mathcal{P}$ denote the collection of all ordered pairs $(e', e'') \in E^2$ of distinct edges of Γ which have no vertex in common.

For each embedding $\mathbf{x} : V \to \mathbb{R}^3$ of $\mathcal{T}_\Gamma$ in space and each pair $\xi := (e', e'') \in \mathcal{P}$, let $\vec{\mathbf{w}}$ denote the *linking vector* connecting the closest points $\mathbf{a} \in \mathbf{x}(e')$ and $\mathbf{b} \in \mathbf{x}(e'')$ on the generalized segments $\mathbf{x}(e')$ and $\mathbf{x}(e'')$ in that order (These may be segments, half-lines, or lines, allowing also edges of infinite length). Since $\mathbf{x}$ is an embedding, $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$. We define an invariant $\phi_\xi(\mathbf{x}) \in \mathbb{Z}/3 = \{-1, 0, 1\}$ by:

$$\phi_\xi(\mathbf{x}) \;:=\; \begin{cases} \mathrm{sgn}(\vec{\mathbf{w}} \cdot (\mathbf{x}(e') \times \mathbf{x}(e''))) & \text{if } \mathbf{a} \text{ is interior to } \mathbf{x}(e'),\ \mathbf{b} \text{ to } \mathbf{x}(e''), \\ & \text{and } \mathbf{x}(e') \text{ and } \mathbf{x}(e'') \text{ are not coplanar} \\ 0 & \text{otherwise.} \end{cases}$$

This is just the *linking number* $\mathrm{lk}_{(\ell', \ell'')}$ of the lines ℓ' and ℓ'' containing $\mathbf{x}(e')$ and $\mathbf{x}(e'')$, respectively (cf. [DV]).

3.2. **Definition.** If we let $\mathcal{F}$ denote the product space $(\mathbb{R}^3)^V \times (\mathbb{Z}/3)^{\mathcal{P}}$, the collection of invariants $\phi_\xi(\mathbf{x})$ together define a (not necessarily continuous) function $\Phi : \mathrm{Emb}(\mathcal{T}_\Gamma) \to \mathcal{F}$, equipped with a projection $\pi : \mathcal{F} \to (\mathbb{R}^3)^V$, such that $\pi \circ \Phi$ is the inclusion $j : \mathrm{Emb}(\mathcal{T}_\Gamma) \hookrightarrow (\mathbb{R}^3)^V$ of §2.1.

We now define an equivalence relation $\sim$ on $\mathcal{F}$ generated as follows: consider a Cauchy sequence $(\mathbf{x}_i)_{i=1}^\infty$ in $X = \mathrm{Emb}(\mathcal{T}_\Gamma)$ with respect to the path metric d_{path} (cf. §2.1). Since $j : \mathrm{Emb}(\mathcal{T}_\Gamma) \hookrightarrow (\mathbb{R}^3)^V$ is an inclusion into a complete metric space, and d_{path} bounds the Euclidean metric in $(\mathbb{R}^3)^V$, the sequence $j(\mathbf{x}_i)_{i=1}^\infty$ converges to a point $\mathbf{x} \in (\mathbb{R}^3)^V$.

If there are two (distinct) sequences $\vec{\alpha} = (\alpha_\xi)_{\xi \in \mathcal{P}}$ and $\vec{\beta} = (\beta_\xi)_{\xi \in \mathcal{P}}$ in $(\mathbb{Z}/3)^{\mathcal{P}}$ and a Cauchy sequence $(\mathbf{x}_i)_{i=1}^\infty$ as above such that for each $N > 0$ there are $m, n \geq N$ with $\phi_\xi(\mathbf{x}_m) = \alpha_\xi$ and $\phi_\xi(\mathbf{x}_n) = \beta_\xi$ for all $\xi \in \mathcal{P}$, then we set $(\mathbf{x}, \vec{\alpha}) \sim (\mathbf{x}, \vec{\beta})$ in $\mathcal{F}$, where $\mathbf{x} = \lim_i j(\mathbf{x}_i)$ in $(\mathbb{R}^3)^V$.

Finally, let $\widetilde{\mathcal{F}} := \mathcal{F}/\sim$ be the quotient space, with $q : \mathcal{F} \to \widetilde{\mathcal{F}}$ the quotient map. Note that the projection $\pi : \mathcal{F} \to (\mathbb{R}^3)^V$ induces a well-defined surjection $\widetilde{\pi} : \widetilde{\mathcal{F}} \to (\mathbb{R}^3)^V$.

Let Σ denote the complement of $\mathcal{C}(\Gamma) \in \mathcal{C}^{im}(\Gamma)$. Generically, there is a dense open subset U of Σ in which any two intersecting edges do so in isolated points internal to both. By Definition 2.3 we have the following:

3.3. **Proposition.** *The function* $\Phi : \mathrm{Emb}(\mathcal{T}_\Gamma) \to \mathcal{F}$ *induces a continuous map* $\widetilde{\Phi} : \widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma) \to \widetilde{\mathcal{F}}$.

3.4. **Definition.** Given an abstract linkage $\mathcal{T}_\Gamma = (V, E)$, the image of the map $\widetilde{\Phi} : \widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma) \to \widetilde{\mathcal{F}}$, denoted by $\underline{\mathrm{Emb}}(\mathcal{T}_\Gamma)$, is called the *blow-up* of the space of embeddings $\mathrm{Emb}(\mathcal{T}_\Gamma)$. It contains the *blow-up* $\underline{\mathcal{C}}(\Gamma)$ of the configuration space $\widehat{\mathcal{C}}(\Gamma)$ as a closed subspace; this is defined to be the closure of the image of $\widetilde{\Phi}|_{\widehat{\mathcal{C}}(\Gamma)}$.

3.5. **Proposition.** *All fibers of* $\widetilde{\pi}$ *are finite, the restriction of* $\underline{\pi}$ *to* $\mathcal{C}(\Gamma)$ *(or* $\mathrm{Emb}(\mathcal{T}_\Gamma)$*) is an embedding, while the restriction of* $\underline{\pi}$ *to* $\underline{\pi}^{-1}(U) \to U$ *is a covering map.*

Proof. Observe that the identifications made by the equivalence relation $\sim$ on $\mathcal{F}$ (or $\mathcal{G}$) do not occur over points of U, since for any $\mathbf{x} \in U$ and $\xi = (e', e'') \in \mathcal{P}$, the intersection of $\mathbf{x}(e')$ and $\mathbf{x}(e'')$ is at a single point internal to both (and in particular, $\mathbf{x}(e')$ and $\mathbf{x}(e'')$ are not parallel). Therefore, for any Cauchy sequence $(\mathbf{x}_i)_{i=1}^{\infty}$ in $\mathrm{Emb}(\mathcal{T}_\Gamma)$ (or $\mathcal{C}(\Gamma)$) converging to $\mathbf{x}$, there is a neighborhood N of $\mathbf{x}$ in $\mathrm{Emb}(\mathcal{T}_\Gamma)$ (or $\mathcal{C}(\Gamma)$) where $\phi_\xi(\mathbf{x}_i)$ is constant $+1$ or constant -1 for all $\xi \in \mathcal{P}$. $\square$

We may summarize our results so far in the following diagram:

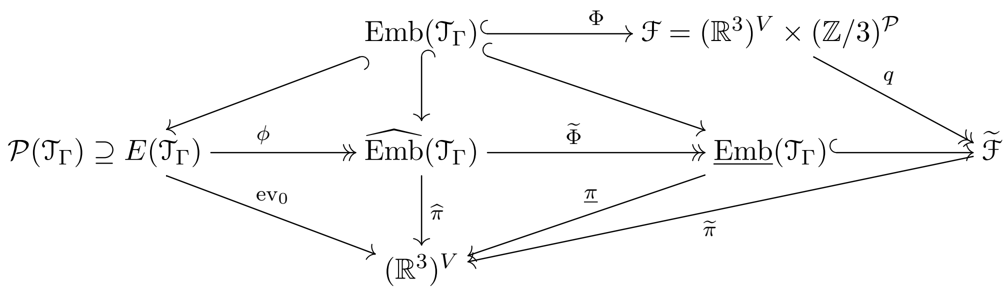

FIGURE 1. Spaces of embeddings

Similarly one may formulate the same diagram for the various types of configuration spaces.
where $\widehat{\pi}$ is generically a surjection (unless $\widehat{\mathcal{C}}(\Gamma)$ has isolated configurations).

As we shall see, the global structure of the blow up $\underline{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ (or $\underline{\mathcal{C}}(\Gamma)$) can be quite involved, even for the simple linkage Γ_2 consisting of two lines. The local structure is also hard to understand. virtual configrations can be any of $k \geq 3$ edges meet at a single point P (k will be called the *multiplicity* of the intersection at P), Three or more edges meet pairwise (or with higher multiplicities), etc.

3.6. **Edge and elbow.** Now consider the case where an interior point of one edge e_1 meets a vertex v common to two other edges e_2' and e_2'' (thus forming an "elbow" Λ_2), as in Figure 2:

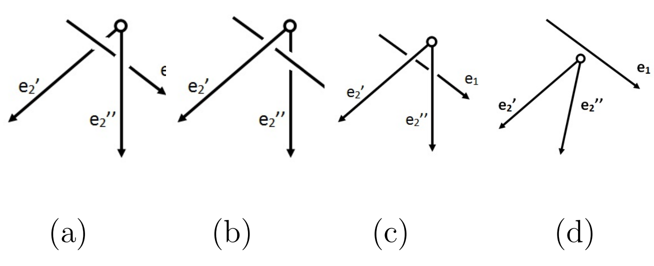

FIGURE 2. Edge and elbow embedding

We can think of $\{e_1, \Lambda\}$ as forming a (disconnected) abstract linkage $\mathcal{T}_\Gamma = (V, E)$, so as in §3.1, for each embedding $\mathbf{x} : V \to \mathbb{R}^3$ of $\mathcal{T}_\Gamma$ in space we have two invariants $\phi_{\xi'}(\mathbf{x}), \phi_{\xi''}(\mathbf{x}) \in \mathbb{Z}/3 = \{-1, 0, 1\}$ – namely, the linking numbers of $\mathbf{x}(e_1)$ with $\mathbf{x}(e_2')$ and of $\mathbf{x}(e_1)$ with $\mathbf{x}(e_2'')$, respectively. Together they yield $\vec{\phi}(\mathbf{x}) \in (\mathbb{Z}/3)^{\mathcal{P}} = \mathbb{Z}/3 \times \mathbb{Z}/3$.

For example, if in the embedding $\mathbf{x}$ shown in Figure 2(a) we have chosen the orientation for $\mathbb{R}^3$ so as to have linking numbers $\vec{\phi}(\mathbf{x}) = (+1, -1)$, say, then Figure 2(b) will have $\vec{\phi}(\mathbf{x}) = (-1, +1)$.

On the other hand, for the embedding of Figure 2(c) we have $\vec{\phi}(\mathbf{x}) = (-1,-1)$, while for Figure 2(d) we have $\vec{\phi}(\mathbf{x}) = (0,0)$, since the nearest points to $\mathbf{x}(e_1)$ on $\mathbf{x}(e'_2)$ or $\mathbf{x}(e''_2)$ are not interior points of the latter.

Thus we see that the immersed configuration represented by Figure 3(a), in which $\mathbf{x}(e_1)$ passes through the vertex $\mathbf{x}(v)$, but is not coplanar with $\mathbf{x}(e'_2)$ and $\mathbf{x}(e''_2)$, has two preimages in the blowup $\underline{\mathrm{Emb}}(\mathcal{T}_\Gamma) = \underline{\mathcal{C}}(\Gamma)$, one of which corresponds to Figure 2(a) (with invariants $(+1,+1)$), while the other preimage corresponds to both Figures 2(c)-(d), under the equivalence relation of §3.2, since we can have Cauchy sequences of either type converging to 3(a).

On the other hand, the immersed configuration represented by Figure 3(b), in which $\mathbf{x}(e_1)$ passes through the vertex $\mathbf{x}(v)$, and all edges are coplanar, is represented by three distinct types of inequivalent Cauchy sequences in $\mathcal{C}(\Gamma)$, corresponding to Figure 2(a), Figure 2(b), and Figure 2(c)-(d), respectively. Thus it has *three* preimages in the blowup $\underline{\mathrm{Emb}}(\mathcal{T}_\Gamma) = \underline{\mathcal{C}}(\Gamma)$. This is the reason we used invariants in $\mathbb{Z}/3$, rather than $\mathbb{Z}/2$.

(a) (b)

FIGURE 3. Virtual edge and elbow configuration

One further situation we must consider in analyzing the edge-elbow linkage is when two or more edges coincide:

(a) When only $\mathbf{x}(e'_2)$ and $\mathbf{x}(e''_2)$ coincide – that is, the elbow is closed – we still have the two cases described in Figure 3.

(b) If $\mathbf{x}(e_1)$ coincides with $\mathbf{x}(e'_2)$, say, with $\mathbf{x}(v)$ internal to $\mathbf{x}(e_1)$, the pre-image in $\underline{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ is a single virtual configuration (since all cases are identified under $\sim$). This is true whether or not the elbow is closed.

3.7. *Remark.* One may show that for all types of pairs of generalized segmentssegments instead of intervals the maps $\widetilde{\Phi} : \widehat{\mathrm{Emb}}(\mathcal{T}_\Gamma) \to \underline{\mathrm{Emb}}(\mathcal{T}_\Gamma)$ and $\widetilde{\Phi} : \widehat{\mathcal{C}}(\Gamma) \to \underline{\mathcal{C}}(\Gamma)$ of Figure 1 are homeomorphisms – that is, the completed configuration space is identical with the blow-up.

4. THE GLOBAL STRUCTURE

In order to deal with more complex virtual configurations, we need to understand the completed configuration spaces of n lines in $\mathbb{R}^3$.

The configuration space of (non-intersecting) skew lines in $\mathbb{R}^3$ have been studied from several points of view (see [CP, P] and the surveys in [DV]). However, here we are mainly interested in the completion of this space, and in particular in the virtual configurations where the lines "intersect" (but still retain the information on their mutual position, in some sense).

4.1. **Definition.** Let $\mathcal{L}$ denote the space of oriented lines in $\mathbb{R}^3$, and $\mathcal{N}$ the space of *all* lines in $\mathbb{R}^3$ (so we have a double cover $\mathcal{L} \twoheadrightarrow \mathcal{N}$).

An oriented line is determined by a choice of a basepoint in $\mathbb{R}^3$ and a vector in S^2, and since the basepoint is immaterial, the space $\mathcal{L}$ of all oriented lines in $\mathbb{R}^3$ is

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$(\mathbb{R}^3 \times S^2)/\mathbb{R}$, where $\mathbb{R}$ acts by translation of the basepoint along ℓ. Alternatively, we can associate to each oriented line ℓ the pair $(\vec{\mathbf{v}}, \mathbf{x})$, where $\vec{\mathbf{v}} \in S^2$ is the unit direction vector of ℓ, and $\mathbf{x} \in \mathbb{R}^3$ is the nearest point to the origin on ℓ, allowing us to identify:

$$\mathcal{L} \;\cong\; \{(\vec{\mathbf{v}}, \mathbf{x}) \in S^2 \times \mathbb{R}^3 \;:\; \mathbf{x} \cdot \vec{\mathbf{v}} = 0\}\ . \tag{4.2}$$

If $\Gamma_n = \mathcal{T}_{\Gamma_n}$ is the linkage consisting of n oriented lines, we thus have an isometric embedding π of $\mathcal{C}(\Gamma_n) = \operatorname{Emb}(\mathcal{T}_{\Gamma_n})$ in the product $\mathcal{L}^{\times n}$, which extends to a surjection $\hat{\pi} : \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n}) \to \mathcal{L}^{\times n}$ (no longer one-to-one).

Denote by $\mathcal{L}_{\mathbf{0}}^{\times n}$ the subspace of $\mathcal{L}^{\times n}$ consisting of all those lines passing through the origin (that is, with $\mathbf{x} = \vec{\mathbf{0}}$), so $\mathcal{L}_{\mathbf{0}}^{\times n} \cong (S^2)^n$, and let $\widehat{\mathcal{E}}_n := \hat{\pi}^{-1}(\mathcal{L}_{\mathbf{0}}^{\times n})$. Thus $\widehat{\mathcal{E}}_n$ is a *virtual configuration space* of n lines passing through the origin.

We denote by Σ the subspace of $\mathcal{L}^{\times n} \subseteq (S^2 \times \mathbb{R}^3)^n$ for which at least two of the n unit vectors in S^2 are parallel:

$$\Sigma \;:=\; \{(\vec{\mathbf{v}}_1, \mathbf{x}_1 \dots, \vec{\mathbf{v}}_n, \mathbf{x}_n) \in S^2)^n \;:\; \exists 1 \leq i < j \leq n\ \exists 0 \neq \lambda \in \mathbb{R},\ \vec{\mathbf{v}}_i = \lambda \vec{\mathbf{v}}_j\}\ .$$

Finally, let $\hat{\Sigma} := \hat{\pi}^{-1}(\Sigma)$ denote the corresponding *singular subspace* of $\widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n})$.

4.3. **Proposition.** *There is a deformation retract* $\rho : \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n}) \to \widehat{\mathcal{E}}_n$.

Proof. For each $t \in [0,1]$ we may use (4.2) to define a map $h_t : \mathcal{L}^{\times n} \to \mathcal{L}^{\times n}$ by setting

$$h_t((\vec{\mathbf{v}}_1, \mathbf{x}_1), \dots, (\vec{\mathbf{v}}_n, \mathbf{x}_n)) \;:=\; ((\vec{\mathbf{v}}_1, t\mathbf{x}_1), \dots, (\vec{\mathbf{v}}_n, t\mathbf{x}_n))\ .$$

For $t > 0$, h_t is equivalent to applying the t-dilitation about the origin in $\mathbb{R}^3$ to each line in $\mathcal{T}_{\Gamma_n}$. Thus it takes the subspace $\operatorname{Emb}(\mathcal{T}_{\Gamma_n})$ of $\mathcal{L}^{\times n}$ to itself, and therefore extends to a map $\widehat{h}_t : \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n}) \to \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n})$.

Now consider a Cauchy sequence $\{P^{(i)}\}_{i=0}^{\infty}$ in $\operatorname{Emb}(\mathcal{T}_{\Gamma_n})$, of the form

$$P^{(i)} \;=\; ((\vec{\mathbf{v}}_1^{(i)}, \mathbf{x}_1^{(i)}), \dots, (\vec{\mathbf{v}}_n^{(i)}, \mathbf{x}_n^{(i)}))\ , \tag{4.4}$$

converging to a virtual configuration $P \in \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n})$. Choosing any sequence $(t_i)_{i=0}^{\infty}$ in $(0,1]$ converging to 0, we obtain a new Cauchy sequence $\{h_{t_i}(P^i)\}_{i=0}^{\infty}$ with

$$h_{t_i}(P^{(i)}) \;=\; ((\vec{\mathbf{v}}_1^{(i)}, t_i\mathbf{x}_1^{(i)}), \dots, (\vec{\mathbf{v}}_n^{(i)}, t_i\mathbf{x}_n^{(i)}))\ ,$$

which is still a Cauchy sequence in $\operatorname{Emb}(\mathcal{T}_{\Gamma_n})$, and furthermore $\lim_{i\to\infty} t_i\mathbf{x}_j^{(i)} = \mathbf{0}$ for all $1 \leq j \leq n$ since the vectors $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ have a common bound K for all $i \in \mathbb{N}$. Thus $\{h_{t_i}(P^i)\}_{i=0}^{\infty}$ represents a virtual configuration P' in $\widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n})$ with $\pi(P') \in \mathcal{L}_{\mathbf{0}}^{\times n}$, and thus $P' \in \widehat{\mathcal{E}}_n$. Moreover, choosing a different sequence $(t_i)_{i=0}^{\infty}$ yields the same P'. Thus if we set $\widehat{h}_0(P) := P'$, we obtain the required map $\rho := \widehat{h}_0 : \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n}) \to \widehat{\mathcal{E}}_n$, as well as a homotopy $H : \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n}) \times [0,1] \to \widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n})$ with $H(P,t) = \widehat{h}_t(P)$ for $t > 0$ – and thus $H(-,1) = \operatorname{Id}$ – and $H(-,0) = \rho$. □

4.5. **Corollary.** *The completed configuration space* $\widehat{\operatorname{Emb}}(\mathcal{T}_{\Gamma_n})$ *of n oriented lines in* $\mathbb{R}^3$ *is homotopy equivalent to the completed space* $\widehat{\mathcal{E}}_n$ *of n oriented lines through the origin.*

Corollary 4.5 allows us to reduce the study of the homotopy type of the completed configuration space of n (oriented) lines in $\mathbb{R}^3$ to the that of the simpler subspace of n lines through the origin (where we may fix ℓ_1 to be the x-axis).

4.6. **The case of two lines again.** For $n = 2$, the remaining (oriented) line ℓ_2 is determined by its direction vector $\vec{\mathbf{v}} \in \mathbf{S}^2$, which is aligned with ℓ_1 at the north pole, say, and reverse-aligned at the south pole. Since we need to take into account the linking number $\pm 1 \in \mathbb{Z}/2$ of ℓ_1 and ℓ_2, we actually have two copies of $\mathbf{S}^2$. However, the north and south poles of these spheres, corresponding to the cases when ℓ_2 is aligned or reverse-aligned with ℓ_1, must be identified as in Figure 4, so we see that $\widehat{\mathcal{E}}_2 \simeq \mathbf{S}^2 \vee \mathbf{S}^2 \vee \mathbf{S}^1$, .

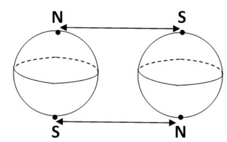

FIGURE 4. The completed configuration space $\widehat{\mathcal{E}}_2$

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