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# Geometric Adaptive Control for Aerial Transportation of a Rigid Body

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**Abstract**

This paper is focused on tracking control for a rigid body payload, that is connected to an arbitrary number of quadrotor unmanned aerial vehicles via rigid links. A geometric adaptive controller is constructed such that the payload asymptotically follows a given desired trajectory for its position and attitude in the presence of uncertainties. The coupled dynamics between the rigid body payload, links, and quadrotors are explicitly incorporated into control system design and stability analysis. These are developed directly on the nonlinear configuration manifold in a coordinate-free fashion to avoid singularities and complexities that are associated with local parameterizations.

## 1. Introduction

By utilizing the high thrust-to-weight ratio, quadrotor unmanned aerial vehicles have been envisaged for aerial load transportation. Most of the existing results for the control of quadrotors to transport a cable-suspended payload are based on the assumption that the dynamics of the payload is decoupled from the dynamics of quadrotors. For example, the effects of the payload are considered as arbitrary external force and torque exerted to quadrotors in Michael et. al (2011). As such, these results may not be suitable for agile load transportation where the motion of cable and payload should be actively suppressed.

Recently, the full dynamic model for an arbitrary number of quadrotors transporting a payload are developed, and based on that, geometric tracking controllers are constructed in an intrinsic fashion. In particular, autonomous transportation of a point mass connected to quadrotors via rigid links is developed in Lee et. al (2013). It has been generalized into a realistic dynamic model that considers the deformation of cables in Goodarzi et. al (2014), and also the attitude dynamics of a payload, that is considered as a rigid body instead of a point mass, is incorporated in Lee (2014). However, these results are based on the assumption that the exact properties of the quadrotors and the payload are available, and that there are no external disturbances, thereby making it challenging to implement those results in actual hardware systems.

The objective of this paper is to construct a control system for an arbitrary number of quadrotors connected to a rigid body payload via rigid links with explicit consideration on uncertainties. In contrast to the work by Lee (2014), unknown disturbance force and moments are assumed to be acting on the payload and the quadrotors. A geometric nonlinear adaptive control system is designed such that both the position and the attitude of the payload asymptotically follow their desired trajectories, while maintaining a certain formation of quadrotors relative to the payload. Including the adaptive control terms requires nontrivial additional developments in the control system design and stability proof, as well as incorporating finite-time stability theory.

The unique property is that the coupled dynamics of the payload, the cables, and

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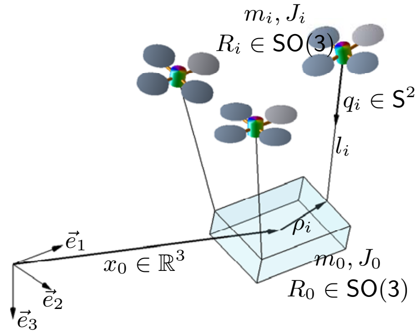


FIGURE 1. Dynamics model: $n$ quadrotors are connect to a rigid body $m_0$ via massless links $l_i$. The configuration manifold is $\mathbb{R}^3 \times \mathsf{SO}(3) \times (\mathsf{S}^2 \times \mathsf{SO}(3))^n$.

quadrotors are explicitly incorporated in control system design for agile load transportations where the motion of the payload relative to the quadrotors are excited nontrivially. Another distinct feature is that the equations of motion and the control systems are developed directly on the nonlinear configuration manifold intrinsically. Therefore, singularities of local parameterization are completely avoided.

As such, the proposed control system is particularly useful for rapid and safe payload transportation in complex terrain, where the position and attitude of the payload should be controlled concurrently. Most of the existing control systems of aerial load transportation suffer from limited agility as they are based on reactive assumptions that ignore the inherent complexities in the dynamics of aerial load transportation. The proposed control system explicitly integrates the comprehensive dynamic characteristics to achieve extreme maneuverability in aerial load transportation. Due to the page limit, proofs and numerical examples are relegated to Lee (2015).

## 2. Problem Formulation

Consider $n$ quadrotor UAVs that are connected to a payload, that is modeled as a rigid body, via massless links (see Figure 1). Throughout this paper, the variables related to the payload is denoted by the subscript 0, and the variables for the $i$-th quadrotor are denoted by the subscript $i$, which is assumed to be an element of $\mathcal{I} = \{1, \cdots n\}$ if not specified. We choose an inertial reference frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and body-fixed frames $\{\vec{b}_{j_1}, \vec{b}_{j_2}, \vec{b}_{j_3}\}$ for $0 \leq j \leq n$ as follows. For the inertial frame, the third axis $\vec{e}_3$ points downward along the gravity and the other axes are chosen to form an orthonormal frame.

The location of the mass center of the payload is denoted by $x_0 \in \mathbb{R}^3$, and its attitude is given by $R_0 \in \mathsf{SO}(3)$, where the special orthogonal group is defined by $\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, R^T R = I,\, \det[R] = 1\}$. Let $\rho_i \in \mathbb{R}^3$ be the point on the payload where the $i$-th link is attached, and it is represented with respect to the zeroth body-fixed frame. The other end of the link is attached to the mass center of the $i$-th quadrotor. The direction of the link from the mass center of the $i$-th quadrotor toward the payload is defined by the unit-vector $q_i \in \mathsf{S}^2$, where $\mathsf{S}^2 = \{q \in \mathbb{R}^3 \,|\, \|q\| = 1\}$, and the length of the $i$-th link is denoted by $l_i \in \mathbb{R}$. Let $x_i \in \mathbb{R}^3$ be the location of the mass center of the $i$-th quadrotor with respect to the inertial frame. As the link is assumed to be rigid, we have $x_i = x_0 + R_0\rho_i - l_i q_i$. The attitude of the $i$-th quadrotor is defined by $R_i \in \mathsf{SO}(3)$, which represents the linear transformation of the representation of a vector from the $i$-th body-fixed frame to the inertial frame. In short, the configuration manifold is $\mathsf{Q} = \mathbb{R}^3 \times \mathsf{SO}(3) \times (\mathsf{S}^2 \times \mathsf{SO}(3))^n$.

The mass and the inertia matrix of the payload are denoted by $m_0 \in \mathbb{R}$ and $J_0 \in \mathbb{R}^{3\times 3}$, respectively. The dynamic model of each quadrotor is identical to Lee et. al (2010). The mass and the inertia matrix of the $i$-th quadrotor are denoted by $m_i \in \mathbb{R}$ and $J_i \in \mathbb{R}^{3\times3}$, respectively. The $i$-th quadrotor can generates a thrust $-f_i R_i e_3 \in \mathbb{R}^3$ with respect to the inertial frame, where $f_i \in \mathbb{R}$ is the total thrust magnitude and $e_3 = [0,0,1]^T \in \mathbb{R}^3$. It also generates a moment $M_i \in \mathbb{R}^3$ with respect to its body-fixed frame. The control input of this system corresponds to $\{f_i, M_i\}_{1\leq i\leq n}$.

In this paper, the external disturbances are modeled as follows. The disturbance force and moment acting on the payload, namely $\Delta_{x_0}, \Delta_{R_0} \in \mathbb{R}^3$ are expressed as

$$\Delta_{x_0} = \Phi_{x_0}(t, \mathsf{q}, \dot{\mathsf{q}})\theta_{x_0}, \quad \Delta_{R_0} = \Phi_{R_0}(t, \mathsf{q}, \dot{\mathsf{q}})\theta_{R_0}, \tag{2.1}$$

where $\Phi_{x_0}, \Phi_{R_0} : \mathbb{R} \times \mathsf{TQ} \to \mathbb{R}^{3\times n_\theta}$ denote matrix-valued, known function of the time $t$ and the tangent vector $(\mathsf{q}, \dot{\mathsf{q}}) \in \mathsf{T_q Q}$ of the configuration manifold, i.e., $\mathsf{q} = (x_0, R_0, q_1, \ldots, q_n, R_1, \ldots R_n)$, and $\theta_{x_0}, \theta_{R_0} \in \mathbb{R}^{n_\theta\times1}$ are fixed, unknown parameters for some $n_\theta$. Similarly, the disturbance force and moment acting on the $i$-th quadrotors are given by

$$\Delta_{x_i} = \Phi_{x_i}(t, \mathsf{q}, \dot{\mathsf{q}})\theta_{x_i}, \quad \Delta_{R_i} = \Phi_{R_i}(t, \mathsf{q}, \dot{\mathsf{q}})\theta_{R_i}, \tag{2.2}$$

where $\Phi_{x_i}, \Phi_{R_i} : \mathbb{R} \times \mathsf{TQ} \to \mathbb{R}^{3\times n_\theta}$ and $\theta_{x_i}, \theta_{R_i} \in \mathbb{R}^{n_\theta\times1}$. Here, the disturbance forces are represented with respect to the inertial frame, and the disturbance moments are represented with respect to the corresponding body-fixed frame.

### 2.1. *Equations of Motion*

The kinematic equations for the payload, quadrotors, and links are given by

$$\dot{q}_i = \omega_i \times q_i = \hat{\omega}_i q_i, \tag{2.3}$$

$$\dot{R}_0 = R_0\hat{\Omega}_0, \quad \dot{R}_i = R_i\hat{\Omega}_i, \tag{2.4}$$

where $\omega_i \in \mathbb{R}^3$ is the angular velocity of the $i$-th link, satisfying $q_i \cdot \omega_i = 0$, and $\Omega_0$ and $\Omega_i \in \mathbb{R}^3$ are the angular velocities of the payload and the $i$-th quadrotor expressed with respect to its body-fixed frame, respectively. The *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined by the condition that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$, and the inverse of the hat map is denoted by the *vee map* $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$.

The equations of motion can be derived according to Lagrangian mechanics as

$$M_q(\ddot{x}_0 - ge_3) - \sum_{i=1}^n m_i q_i q_i^T R_0\hat{\rho}_i\dot{\Omega}_0 = \Delta_{x_0} + \sum_{i=1}^n u_i^{\parallel} + \Delta_{x_i}^{\parallel} - m_i l_i \|\omega_i\|^2 q_i - m_i q_i q_i^T R_0 \hat{\Omega}_0^2 \rho_i, \tag{2.5}$$

$$\begin{aligned}(J_0 - \sum_{i=1}^n m_i\hat{\rho}_i R_0^T q_i q_i^T R_0\hat{\rho}_i)\dot{\Omega}_0 + \sum_{i=1}^n m_i\hat{\rho}_i R_0^T q_i q_i^T(\ddot{x}_0 - ge_3) + \hat{\Omega}_0 J_0\Omega_0 = \Delta_{R_0}\\ + \sum_{i=1}^n \hat{\rho}_i R_0^T(u_i^{\parallel} + \Delta_{x_i}^{\parallel} - m_i l_i\|\omega_i\|^2 q_i - m_i q_i q_i^T R_0\hat{\Omega}_0^2\rho_i),\end{aligned} \tag{2.6}$$

$$\dot{\omega}_i = \frac{1}{l_i}\hat{q}_i(\ddot{x}_0 - ge_3 - R_0\hat{\rho}_i\dot{\Omega}_0 + R_0\hat{\Omega}_0^2\rho_i) - \frac{1}{m_i l_i}\hat{q}_i(u_i^{\perp} + \Delta_{x_i}^{\perp}), \tag{2.7}$$

$$J_i\dot{\Omega}_i + \Omega_i \times J_i\Omega_i = M_i + \Delta_{R_i}, \tag{2.8}$$

where $M_q = m_y I + \sum_{i=1}^n m_i q_i q_i^T \in \mathbb{R}^{3\times3}$, which is symmetric, positive-definite (Lee (2015)).

Recall the vector $u_i \in \mathbb{R}^3$ represents the control force at the $i$-th quadrotor, i.e., $u_i = -f_i R_i e_3$. The vectors $u_i^{\parallel}$ and $u_i^{\perp} \in \mathbb{R}^3$ denote the orthogonal projection of $u_i$ along

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$q_i$, and the orthogonal projection of $u_i$ to the plane normal to $q_i$, respectively, i.e.,

$$u_i^{\parallel} = q_i q_i^T u_i, \quad u_i^{\perp} = -\hat{q}_i^2 u_i = (I - q_i q_i^T) u_i. \tag{2.9}$$

Therefore, $u_i = u_i^{\parallel} + u_i^{\perp}$. In this paper, the subscripts $\parallel$ and $\perp$ denote the component of the vector that is parallel to $q_i$ and the other component of the vector that is perpendicular to $q_i$. Similarly, the disturbance force at the $i$-th quadrotor is decomposed as

$$\Delta_{x_i}^{\parallel} = q_i q_i^T \Phi_{x_i} \theta_{x_i} \triangleq \Phi_{x_i}^{\parallel} \theta_{x_i}, \quad \Delta_{x_i}^{\perp} = (I - q_i q_i^T) \Phi_{x_i} \theta_{x_i} \triangleq \Phi_{x_i}^{\perp} \theta_{x_i}. \tag{2.10}$$

### 2.2. *Tracking Problem*

Define a fixed matrix $\mathcal{P} \in \mathbb{R}^{6\times 3n}$ as

$$\mathcal{P} = \begin{bmatrix} I_{3\times3} & \cdots & I_{3\times3} \\ \hat{\rho}_1 & \cdots & \hat{\rho}_n \end{bmatrix}. \tag{2.11}$$

Assume the links are attached to the payload such that $\mathrm{rank}[\mathcal{P}] \geq 6$. This is to guarantee that there exist enough degrees of freedom in control inputs for both the translational motion and the rotational maneuver of the payload. This requires that the number of quadrotor is at least three, i.e., $n \geq 3$.

It is also assumed that the bounds of the disturbance forces and moments are available, i.e., for known positive constant $B_\Phi, B_\theta \in \mathbb{R}$, we have

$$\max\{\|\Phi_{x_0}\|, \|\Phi_{R_0}\|, \|[\Phi_{x_1}\|, \ldots, \|\Phi_{x_n}\|, \|\Phi_{R_0}\|, \ldots, \|\Phi_{R_n}\|\} < B_\Phi, \tag{2.12}$$

$$\max\{\|\theta_{x_0}\|, \|\theta_{R_0}\|, \|[\theta_{x_1}\|, \ldots, \|\theta_{x_n}\|, \|\theta_{R_0}\|, \ldots, \|\theta_{R_n}\|\} < B_\theta. \tag{2.13}$$

Suppose that the desired trajectories for the position and the attitude of the payload are given as smooth functions of time, namely $x_{0_d}(t) \in \mathbb{R}^3$ and $R_{0_d}(t) \in \mathsf{SO(3)}$. We wish to design a control input of each quadrotor $\{f_i, M_i\}_{1\leq i\leq n}$ such that the tracking errors asymptotically converge to zero along the solution of the controlled dynamics.

## 3. Control System Design For Simplified Dynamic Model

In this section, we consider a simplified dynamic model where the attitude dynamics of each quadrotor is ignored, and we design a control input by assuming that the thrust at each quadrotor, namely $u_i$ can be arbitrarily chosen. The effects of the attitude dynamics of quadrotors will be incorporated in the next section.

In the simplified dynamic model given by (2.5)-(2.7), the dynamics of the payload are affected by the parallel components $u_i^{\parallel}$ of the thrusts, and the dynamics of the links are directly affected by the normal components $u_i^{\perp}$ of the thrusts. This structure motivates the following control system design procedure: first, the parallel components $u_i^{\parallel}$ are chosen such that the payload follows the desired position and attitude trajectory while yielding the desired direction of each link, namely $q_{i_d} \in \mathsf{S}^2$; next, the normal components $u_i^{\perp}$ are designed such that the actual direction of the links $q_i$ follows the desired direction $q_{i_d}$.

### 3.1. *Design of Parallel Components*

Let $a_i \in \mathbb{R}^3$ be the acceleration of the point on the payload where the $i$-th link is attached, that is measured relative to the gravitational acceleration:

$$a_i = \ddot{x}_0 - g e_3 + R_0 \hat{\Omega}_0^2 \rho_i - R_0 \hat{\rho}_i \dot{\Omega}_0. \tag{3.1}$$

The parallel component of the control input is chosen as

$$u_i^{\parallel} = \mu_i + m_i l_i \|\omega_i\|^2 q_i + m_i q_i q_i^T a_i, \tag{3.2}$$

where $\mu_i \in \mathbb{R}^3$ is a virtual control input that is designed later, with a constraint that $\mu_i$ is parallel to $q_i$. Note that the expression of $u_i^{\parallel}$ is guaranteed to be parallel to $q_i$ due to the projection operator $q_i q_i^T$ at the last term of the right-hand side of the above expression.

The motivation for the proposed parallel components becomes clear if (3.2) is substituted into (2.5)-(2.6) and rearranged to obtain

$$m_0(\ddot{x}_0 - ge_3) = \Delta_{x_0} + \sum_{i=1}^{n}(\mu_i + \Delta_{x_i}^{\parallel}), \tag{3.3}$$

$$J_0\dot{\Omega}_0 + \hat{\Omega}_0 J_0 \Omega_0 = \Delta_{R_0} + \sum_{i=1}^{n} \hat{\rho}_i R_0^T(\mu_i + \Delta_{x_i}^{\parallel}). \tag{3.4}$$

Therefore, considering a free-body diagram of the payload, the virtual control input $\mu_i$ corresponds to the force exerted to the payload by the $i$-link, or the tension of the $i$-th link in the absence of disturbances.

Next, we determine the virtual control input $\mu_i$. As in Lee et. al (2010), define position, attitude, and angular velocity tracking error vectors $e_{x_0}, e_{R_0}, e_{\Omega_0} \in \mathbb{R}^3$ for the payload as

$$e_{x_0} = x_0 - x_{0_d}, \quad e_{R_0} = \frac{1}{2}(R_{0_d}^T R_0 - R_0^T R_{0_d})^{\vee}, \quad e_{\Omega_0} = \Omega_0 - R_0^T R_{0_d}\Omega_{0_d}.$$

The desired resultant control force $F_d \in \mathbb{R}^3$ and moment $M_d \in \mathbb{R}^3$ on the payload are

$$F_d = m_0(-k_{x_0}e_{x_0} - k_{\dot{x}_0}\dot{e}_{x_0} + \ddot{x}_{0_d} - ge_3) - \Phi_{x_0}\bar{\theta}_{x_0} - \sum_{i=1}^{n}\Phi_{x_i}^{\parallel}\bar{\theta}_{x_i}, \tag{3.5}$$

$$\begin{aligned} M_d = &-k_{R_0}e_{R_0} - k_{\Omega_0}e_{\Omega_0} + (R_0^T R_{0_d}\Omega_{0_d})^{\wedge} J_0 R_0^T R_{0_d}\Omega_{0_d} \\ &+ J_0 R_0^T R_{0_d}\dot{\Omega}_{0_d} - \Phi_{R_0}\bar{\theta}_{R_0} - \sum_{i=1}^{n}\hat{\rho}_i R_0 \Phi_{x_i}^{\parallel}\bar{\theta}_{x_i}, \end{aligned} \tag{3.6}$$

for positive constants $k_{x_0}, k_{\dot{x}_0}, k_{R_0}, k_{\Omega_0} \in \mathbb{R}$. Here, the estimates of the unknown parameters $\theta_{x_0}, \theta_{x_i}, \theta_{R_0}$ are denoted by $\bar{\theta}_{x_0}, \bar{\theta}_{x_i}, \bar{\theta}_{R_0} \in \mathbb{R}^{n_\theta}$.

These are the ideal resultant force and moment to achieve the control objectives. One may try to choose the virtual control input $\mu_i$ by making the expressions in the right-hand sides of (3.3) and (3.4), namely $\sum_i \mu_i$ and $\sum_i \hat{\rho}_i R_0^T \mu_i$, become identical to $F_d$ and $M_d$, respectively. But, this is not valid in general, as each $\mu_i$ is constrained to be parallel to $q_i$. Instead, we choose the desired value of $\mu_i$, without any constraint, such that

$$\sum_{i=1}^{n}\mu_{i_d} = F_d, \quad \sum_{i=1}^{n}\hat{\rho}_i R_0^T \mu_{i_d} = M_d. \tag{3.7}$$

As rank$[\mathcal{P}] \geq 6$, there exists at least one solution $\mu_{i_d}$ to the above equation for any $F_d, M_d$. Here, we find the minimum-norm solution given by

$$\begin{bmatrix}\mu_{1_d}\\ \vdots \\ \mu_{n_d}\end{bmatrix} = \mathrm{diag}[R_0, \cdots R_0]\, \mathcal{P}^T(\mathcal{P}\mathcal{P}^T)^{-1}\begin{bmatrix}R_0^T F_d \\ M_d\end{bmatrix}. \tag{3.8}$$

The virtual control input $\mu_i$ is selected as the projection of its desired value $\mu_{i_d}$ along $q_i$,

$$\mu_i = (\mu_{i_d}\cdot q_i)q_i = q_i q_i^T \mu_{i_d}, \tag{3.9}$$

and the desired direction of each link, namely $q_{i_d} \in \mathsf{S}^2$ is defined as

$$q_{i_d} = -\frac{\mu_{i_d}}{\|\mu_{i_d}\|}. \tag{3.10}$$

It is straightforward to verify that when $q_i = q_{i_d}$, the resultant force and moment acting on the payload become identical to their desired values.

### 3.2. *Design of Normal Components*

Substituting (3.1) into (2.7) and using (2.10),

$$\dot{\omega}_i = \frac{1}{l_i}\hat{q}_i a_i - \frac{1}{m_i l_i}\hat{q}_i(u_i^\perp + \Delta_{x_i}^\perp). \tag{3.11}$$

Here, the normal component of the control input $u_i^\perp$ is chosen such that $q_i \to q_{i_d}$ as $t \to \infty$. Here, we adopt the control system on the two-sphere studied in Wu (2012), and we augment it with an adaptive control term to handle the disturbance $\Delta_{x_i}^\perp$.

For the given desired direction of each link, its desired angular velocity is obtained from the kinematics equation as $\omega_{i_d} = q_{i_d} \times \dot{q}_{i_d}$. Define the direction and the angular velocity tracking error vectors for the $i$-th link, namely $e_{q_i}, e_{\omega_i} \in \mathbb{R}^3$ as

$$e_{q_i} = q_{i_d} \times q_i, \quad e_{\omega_i} = \omega_i + \hat{q}_i^2 \omega_{i_d}.$$

For positive $k_q, k_\omega \in \mathbb{R}$, the normal component of the control input is chosen as

$$u_i^\perp = m_i l_i \hat{q}_i\{-k_q e_{q_i} - k_\omega e_{\omega_i} - (q_i \cdot \omega_{i_d})\dot{q}_i - \hat{q}_i^2 \dot{\omega}_d\} - m_i \hat{q}_i^2 a_i - \Phi_{x_i}^\perp \bar{\theta}_{x_i}. \tag{3.12}$$

In short, the control force for the simplified dynamic model is given by

$$u_i = u_i^\parallel + u_i^\perp. \tag{3.13}$$

### 3.3. *Design of Adaptive Law*

Next, we design the adaptive laws to construct the estimates of unknown parameters. The following projection operator (Ioannou and Sung (1995)) is introduced such that the estimated parameters stay in the bound of the true parameters given by (2.13).

$$\Pr(\bar{\theta}, y) = \begin{cases} y & \text{if } \|\bar{\theta}\| < B_\theta \\ & \text{or } \|\bar{\theta}\| = B_\theta \text{ and } \bar{\theta}^T y \leq 0, \\ (I_{n_\theta \times n_\theta} - \frac{1}{\|\bar{\theta}\|^2}\bar{\theta}\bar{\theta}^T)y & \text{otherwise.} \end{cases} \tag{3.14}$$

Using this, the adaptive laws are defined as

$$\dot{\bar{\theta}}_{x_0} = \Pr(\bar{\theta}_{x_0}, y_{x_0}), \quad \dot{\bar{\theta}}_{R_0} = \Pr(\bar{\theta}_{R_0}, y_{R_0}), \quad \dot{\bar{\theta}}_{x_i} = \Pr(\bar{\theta}_{x_i}, y_{x_i}), \tag{3.15}$$

where $y_{x_0}, y_{R_0}, y_{x_i} \in \mathbb{R}^{n_\theta}$ are defined as

$$y_{x_0} = \frac{h_{x_0}}{m_0}\Phi_{x_0}^T(\dot{e}_{x_0} + c_x e_{x_0}), \tag{3.16}$$

$$y_{R_0} = h_{R_0}\Phi_{R_0}^T(e_{\Omega_0} + c_R e_{R_0}), \tag{3.17}$$

$$y_{x_i} = h_{x_i}\Phi_{x_0}^T\big[q_i q_i^T\{\frac{1}{m_0}(\dot{e}_{x_0} + c_x e_{x_0}) - R_0\hat{\rho}_i(e_{\Omega_0} + c_R e_{R_0})\} + \frac{1}{m_i l_i}\hat{q}_i(e_{\omega_i} + c_q e_{q_i})\big], \tag{3.18}$$

for positive constants $c_x, c_R, c_q \in \mathbb{R}$ and adaptive gains $h_{x_0}, h_{R_0}, h_{x_i} \in \mathbb{R}$.

The resulting stability properties are summarized as follows (see Lee (2015) for proof).

Proposition 1. *Consider the simplified dynamic model defined by (2.5)-(2.7). For* *given tracking commands $x_{0_d}, R_{0_d}$, a control input is designed as (3.13)-(3.15). Then, there exist the values of controller gains and controller parameters such that the following properties are satisfied.*

*(i) The zero equilibrium of tracking errors $(e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i})$ and the estimation errors $(\bar{\theta}x_0, \bar{\theta}_{R_0}, \bar{\theta}_{x_i})$ is stable in the sense of Lyapunov.*

*(ii) The tracking errors asymptotically coverage to zero.*

*(iii) The magnitude of the estimated parameters is less than $B_\theta$ always, provided that the magnitude of their initial estimates is less than $B_\theta$.*

## 4. Control System Design for Full Dynamic Model

In this section, we incorporate the attitude dynamics of quadrotors. Specifically, the attitude of each quadrotor is controlled such that the third body-fixed axis becomes parallel to the direction of the ideal control force $u_i$ designed in the previous section within a finite time.

The desired direction of the third body-fixed axis of the $i$-th quadrotor, namely $b_{3_i} \in \mathsf{S}^2$ is given by $b_{3_i} = -\frac{u_i}{\|u_i\|}$. This provides two-dimensional constraint on the three-dimensional desired attitude of each quadrotor, and there remains one degree of freedom. To resolve it, the desired direction of the first body-fixed axis $b_{1_i}(t) \in \mathsf{S}^2$ is introduced as a smooth function of time Lee et. al (2010). This corresponds to controlling the additional one dimensional yawing angle of each quadrotor. Thus, the desired attitude of the $i$-th quadrotor is

$$R_{i_c} = \left[ -\frac{(\hat{b}_{3_i})^2 b_{1_i}}{\|(\hat{b}_{3_i})^2 b_{1_i}\|}, \quad \frac{\hat{b}_{3_i} b_{1_i}}{\|\hat{b}_{3_i} b_{1_i}\|}, \quad b_{3_i} \right].$$

In the prior work described in Lee (2014), the attitude of each quadrotor is controlled such that the equilibrium $R_i = R_{i_c}$ becomes exponentially stable, and the stability of the combined full dynamic model is achieved via singular perturbation theory Khalil (1996). However, we can not follow such approach in this paper, as the presented adaptive control system guarantees only the asymptotical convergence of the tracking error variables due to the disturbances, thereby making is challenging to apply the singular perturbation theory. Here, we design the attitude controller of each quadrotor such that $R_i$ becomes equal to $R_{i_c}$ within a finite time via finite-time stability theory Bhat and Bernstein (2000), Yu et. al (2005).

Define the tracking error vectors $e_{R_i}, e_{\Omega_i} \in \mathbb{R}^3$ as

$$e_{R_i} = \frac{1}{2}(R_{i_c}^T R_i - R_i^T R_{i_c})^\vee, \quad e_{\Omega_i} = \Omega_i - R_i^T R_{i_c} \Omega_{i_c}.$$

The time-derivative of $e_{R_i}$ can be written as Lee et. al (2010)

$$\dot{e}_{R_i} = \frac{1}{2}(\mathrm{tr}\left[R_i^T R_{i_c}\right] I - R_i^T R_{i_c}) e_{\Omega_i} \triangleq E(R_i, R_{i_c}) e_{\Omega_i}. \tag{4.1}$$

For $0 < r < 1$, define $S : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ as

$$S(r, y) = \left[|y_1|^r \mathrm{sgn}(y_1), \quad |y_2|^r \mathrm{sgn}(y_2), \quad |y_3|^r \mathrm{sgn}(y_3)\right]^T,$$

where $y = [y_1, y_2, y_3]^T \in \mathbb{R}^3$, and $\mathrm{sgn}(\cdot)$ denotes the sign function. For positive constants $k_R, l_R$, the terminal sliding surface $s_i \in \mathbb{R}^3$ is designed as

$$s_i = e_{\Omega_i} + k_R e_{R_i} + l_R S(r, e_{R_i}). \tag{4.2}$$

We can show that when confined to the surface of $s_i \equiv 0$, the tracking errors become zero in a finite time. To reach the sliding surface, for positive constants $k_s, l_s$, the control moment is designed as

$$M_i = -k_s s_i - l_s S(r, s_i) + \Omega_i \times J_i\Omega_i - (k_R J_i + l_s r J_i \mathrm{diag}_j[|e_{R_{i_j}}|^{r-1}])E(R_i, R_{c_i})e_{\Omega_i} - J_i(\hat{\Omega}_i R_i^T R_{i_c}\Omega_{i_c} - R_i^T R_{i_c}\dot{\Omega}_{i_c}). \tag{4.3}$$

The thrust magnitude is chosen as the length of $u_i$, projected on to $-R_i e_3$, i.e., $f_i = -u_i \cdot R_i e_3$, which yields that the thrust of each quadrotor becomes equal to its desired value $u_i$ when $R_i = R_{i_c}$. Stability of the corresponding controlled systems for the full dynamic model can be shown by using the fact that the full dynamic model becomes exactly same as the simplified dynamic model within a finite time (see Lee (2015) for proof).

PROPOSITION 2. *Consider the full dynamic model defined by (2.5)-(2.8). For given tracking commands $x_{0_d}, R_{0_d}$ and the desired direction of the first body-fixed axis $b_{1_i}$, control inputs for quadrotors are designed as (4.3) and $f_i = -u_i \cdot R_i e_3$. Then, there exists controller parameters such that the tracking error variables $(e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i})$ asymptotically converge to zero, and the estimation errors are uniformly bounded.*

This implies that the payload asymptotically follows any arbitrary desired trajectory both in translations and rotations in the presence of uncertainties. In contrast to the existing results in aerial transportation of a cable suspended load, it does not rely on any simplifying assumption that ignores the coupling between payload, cable, and quadrotors. Also, the presented global formulation on the nonlinear configuration manifold avoids singularities and complexities that are inherently associated with local coordinates. As such, the presented control system is particularly useful for agile load transportation involving combined translational and rotational maneuvers of the payload in the presence of uncertainties. Numerical examples are available at Lee (2015): `http://arxiv.org/abs/1503.01148` and `http://youtu.be/nOWErfdzZLU`.

## REFERENCES


S. BHAT AND D. BERNSTEIN, *Finite-time stability of continuous autonomous systems*, SIAM Journal of Control and Optimization, 38 (2000), pp. 751–766.

F. GOODARZI, D. LEE, AND T. LEE, *Geometric stabilization of a quadrotor UAV with a payload connected by flexible cable*, American Control Conference, 2014, pp. 4925–4930.

P. IOANNOU AND J. SUNG, *Robust Adaptive Control*, Prentice Hall, 1995.

H. KHALIL, *Nonlinear Systems*, Prentice Hall, 1996.

T. LEE, *Geometric control of multiple quadrotor UAVs transporting a cable-suspended rigid body*, IEEE Conference on Decision and Control, 2014, pp. 6155–6160.

———, *Geometric adaptive control for aerial transportation of a rigid body.* arXiv:150301148, 2015. `http://arxiv.org/abs/1503.01148`

T. LEE, M. LEOK, AND N. MCCLAMROCH, *Geometric tracking control of a quadrotor aerial vehicle on* SE(3), IEEE Conference on Decision and Control, 2010, pp. 5420–5425.

T. LEE, K. SREENATH, AND V. KUMAR, *Geometric control of cooperating multiple quadrotor UAVs with a suspended load*, IEEE Conference on Decision and Control, 2013, pp. 5510–5515.

N. MICHAEL, J. FINK, AND V. KUMAR, *Cooperative manipulation and transportation with aerial robots*, Autonomous Robots, 30 (2011), pp. 73–86.

T. WU, *Spacecraft relative attitude formation tracking on SO(3) based on line-of-sight measurements*, Master's thesis, The George Washington University, 2012.

S. YU, X. YU, B. SHIRINZADEH, AND Z. MAN, *Continuous finite-time control for robotic manipulators with terminal sliding mode*, Automatica, 41 (2005), pp. 1957–1964.