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Time parametrized motion planning

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Abstract

This paper proposes several extensions to the existing method of parametrizing speed along a prescribed path. The velocity is modified by isotropically stretching/shrinking the tangent space. The path in closed form is determined by substitution, without the computational cost of re-integrating the velocity function. This concept can be extended to anisotropic stretching/shrinking of the momentum space to change the direction and magnitude of the momentum vector.

The physical constraints of the actuators on increasing and decreasing momentum (and its differentials) are incorporated into a parametrization function that achieves maximum distance for a given input of energy and satisfies boundary conditions on the momentum.

This method of time parametrization especially applies to Geometric Control, where the Hamiltonian minimizes some cost function and matches the boundary configuration constraints but not the velocity constraints. The optimal (geometric) trajectory is modified by the parametrization so that the cost function is minimized if the stretching is stopped at any time. The forces stretching the momentum space are identifiable from the formulation.

An asymmetric rigid satellite illustrates the modification of angular momentum, with independent parametrizations of the linear momentum.

1. Introduction

The use of autonomous vehicles is becoming increasingly important as technological advances enable them to aid humans in dangerous environments. Computer path planning and control are making automation possible but computationally efficient methods are required, as evidenced by previous efforts by Loizou (2012), Lee *et al.* (2012) and Maclean *et al.* (2011). We consider those optimal trajectories of a predefined vehicle which can be expressed in closed form. The parameter values of the motion are determined to move from the initial configuration to the target configuration in a prespecified time. In this paper, the corresponding optimal momentum is modified using an simple computation to match the starting and finishing conditions and minimize the accelerating and braking forces required. The actual velocity and configuration can be tracked by comparison with the resultant trajectory found by substituting into the original optimal motion without recalculation.

The emphasis of the method is to optimize the trajectory in closed form and to optimize the momentum changes. The requirement for numerical iteration is limited to the single step of matching to the final configuration and this is preformed once. The calculation of the configuration and velocity is done by substitution into the closed form solution.

We extend the idea of modifying speed along a path (as used by Xargay *et al.* (2013) and others) to modifying momentum. Momentum is conserved in the absence of external forces, but by applying forces, the magnitude and direction of momentum can be changed. This change is envisaged as an anisotropic stretching or shrinking of the cotangent space (known as momentum space) in which momentum (for example, linear and/or angular momentum) is represented by a vector.

Biggs & Horri (2012) used Geometric Control theory to determine a trajectory in closed form based for a axially symmetric vehicle in advance. Spindler (1997) used a spherically symmetric vehicle. Numerical methods are required to determine the parameters of the motion (basically the direction of angular momentum) to achieve the boundary configurations. This paper uses an asymmetric vehicle.

A simple time parametrization function is proposed which achieves the final configuration in the shortest possible time, within the constraints of a maximum momentum and maximum derivatives of momentum. The effort required is limited to changing the momentum in the required direction. Other authors have used simple mathematical functions (polynomials Smith (2008) or reciprocal functions Spindler (2002)) or a convolution operator (Yang & Choi (2013)). These provide smooth functions but not the optimal solution as defined in this paper. Pham (2013) suggests a "bang-bang" solution but limits the derivatives considered to one (the rate of change of momentum).

Geometric control theory has been used to tackle motion planning problems for nonholonomic systems on Lie groups. It has the advantage that the motion is always considered from the body frame of reference. The moments of inertia are fixed in the body frame, as are any motors providing thrust or torque. This paper addresses the disadvantage that constraints on momentum (or velocity) are not considered in the basic method.

The paper starts by presenting the idea of using time as a free parameter so that the momentum space is stretched isotropically. The modified motion is determined by substituting the parametrized time function $t = F(\tau)$ into the original motion.The momentum $\mathbf{p}(t)$ becomes $\mathbf{p}(F(\tau))\frac{dF(\tau)}{dt}$. A parametrized time function is formulated which achieves the greatest distance in a fixed time, whilst being constrained by a maximum momentum and maximum rates of change that the actuators can achieve. The differences in maximum rate of increase (related to acceleration) and maximum rate of decrease (deceleration) are incorporated.

A trajectory is broken into motion primitives which increase, maintain or change the momentum. The continuity of the joins between primitives are dependent of the continuity of the original momentum $\mathbf{p}$ and the parametrization function F.

Section 3 considers the rotation of an asymmetric rigid body. The constant angular momentum is expressed as 3 components in the principle body frame. This is integrated to give the free rotation in a nearly closed form. Only the angular velocity about the fixed axis of rotation has to be integrated numerically. Finally the rotation is adjusted to satisfy the momentum boundary conditions by applying a time parametrization function.

In Section 4, an asymmetric satellite is realigned about a fixed axis of rotation, starting with zero angular momentum and reaching a maximum value. The concept of stretching the cotangent space anisotropically is used to maintain the linear momentum is one direction whilst introducing a sideways motion.

2. Time parametrization

In this section, the momentum vector is modified in magnitude

- without imposing the condition that the motion starts at rest and stops at the end of each motion primitive
- incorporating different actuator limits for the increase and decrease of momentum, and their rates of change

2.1. *Background*

In most, if not all, of the previous work on modifying the speed along a path, there is the assumption that the initial speed v is constant in the direction of travel. Spindler (1997) and Chaturvedi *et al.* (2011)) consider trajectories with fixed velocity (in direction and magnitude) and modify it to start at rest and stop at the end. The constant speed is modified by multiplying by a function $f(\tau)$ where τ is the parametrized time and $f(0) = f(T) = 0$ when T is the time at the end of the trajectory. The constant velocities were linear in Smith (2008) and eigenaxis angular velocities in Spindler (1997). Flasskamp & Ober-Blobaum (2012) and Frazzoli *et al.* (2005) worked on joining trim primitives (where the unmodified velocity is fixed) into total trajectories but only by stopping the motion and starting again. In some conditions, this can be avoided.

Lee *et al.* (2012) uses a convolution-based parametrization function which starts and finishes zero, and specifies the maximum velocity, acceleration, jerk and higher differentials. It is symmetrical, with the same maximum rates of change whether increasing or decreasing.

In comparison, Maclean *et al.* (2011) used a feedback controller to achieve the optimal velocity and then reset the target velocity to zero towards the end of the trajectory.

More recently, Xargay *et al.* (2013) adjusted the speed along a prescribed path. They consider a fleet of vehicles. Feedback is used to adjust the speed of each vehicle along its prescribed path so that they avoid each other and achieve joint objectives. Aguiar *et al.* (2008) showed that the speed along a geometric path could be adjusted and that, for some class of systems, performance limitations are avoided. Aguiar *et al.* (2008) showed that, for a geometric path, the original controls keep the system on the path and additional controls manage the speed adjustment.

2.2. *General parametrization*

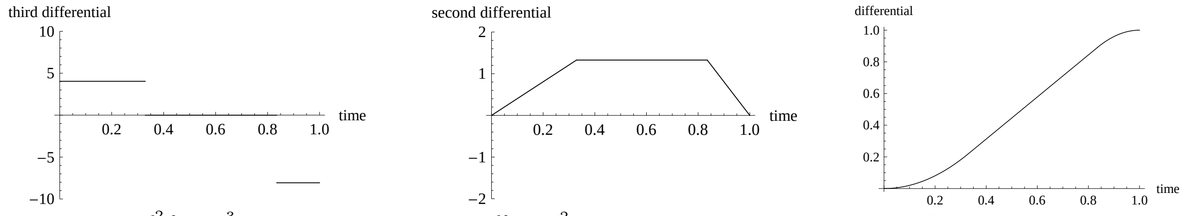
In this paper, momentum is considered as the starting point. It is conserved in the absence of external forces. In Euclidean space, linear and angular momentum are both conserved. In other configuration spaces, the total momentum is conserved. The momentum can be represented by a vector in momentum space; even for multiple bodies. The conserved momentum vector can always be expressed in component terms $\mathbf{p} = \sum_i p_i(t)\, e_i$ in a moving frame of reference (identified by unit vectors $\{e_i\}$).

The function $f(\tau)$ can be considered to stretch and shrink the momentum space evenly in all directions so that $\mathbf{p}(\mathbf{t}) \Longrightarrow \mathbf{p}(\mathbf{F}(\tau))f(\tau)$, where $t = F(\tau)$ and $\frac{d}{d\tau}F = f(\tau)$. The function $f(\tau)$ must be positive but can start and finish at any values to meet specified boundary conditions.

Any path can be represented by a vector $\mathbf{x}(t)$, typically made up of position and attitude components which change over time. In general, $\mathbf{x}(t)$ is any point on the manifold of possible configurations. With the parametrization, the motion becomes

$$\mathbf{x}(t) \Longrightarrow \mathbf{x}(F(\tau)) \equiv \mathbf{x}_f(\tau)$$

If the parametrization function has the property that $T = F(T)$, then the reparameterized motion achieves the same position and attitude as the original motion. That is, $\mathbf{x}(T) = \mathbf{x}_f(T)$. Hence there are periods of time when $f(\tau)$ is increasing (acceleration) and others when it is decreasing (deceleration).



1.a $\frac{d^2 f}{d\tau^2} \equiv \frac{d^3 F}{d\tau^3}$ (cf jerk) 1.b $\frac{df}{d\tau} \equiv \frac{d^2 F}{d\tau^2}$ (cf acceleration) 1.c $f \equiv \frac{dF}{d\tau}$ (cf velocity)

FIGURE 1. Parametrization function and its derivatives

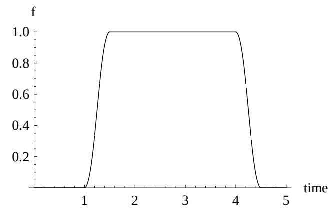


FIGURE 2. Parametrization function $f(\tau)$, joining increasing, constant and decreasing motion primitives

2.3. *Parametrization functions*

Any non-decreasing differentiable function on the time interval $\tau \in [0, T]$ which starts at zero can be used as a parametrization function $F(\tau) = t$.

$$F(0) = 0, \frac{dF}{d\tau} \equiv f(\tau) \geq 0 \tag{2.1}$$

A simple function that fulfills the basic criteria for a parametrization function is based on Heaviside Unit Step function - see Figure 1.a. The resulting parametrization function achieves the maximum possible distance in any time period, within the constraints of maximum velocity, maximum acceleration/braking and rate of increase/decrease of these two.

The third derivative of parametrization function has positive, zero and negative values reflecting the acceptable/manageable maximum jerk at the start and end of the acceleration stage - see Figure 1.a. This applies until the acceleration has reached its achievable maximum (determined experimentally).

The maximum acceleration is applied and then reduced to zero as rapidly as possible so that the maximum momentum is reached as acceleration stops - see Figure 1.b.

A similar decelerating parametrization function considers the maximum braking effort, and the maximum rate of increasing and decreasing the braking.

2.4. *Joining Motion Primitives*

A complex trajectory can be built up from separate motion primitives. Each motion primitive concentrates on a few objectives; for example, increasing, maintaining or decreasing the momentum. This enables easier consideration of the different mechanisms (and parameters) to achieve each objective. To enable no change in acceleration at the joins with the previous and succeeding motions, the second and third differentials of the parametrization function are zeroed at the start and finish of the motion

$$\frac{d^2}{dt^2} F(0) = \frac{d^2}{dt^2} F(T) = 0 \,\text{and}\, \frac{d^3}{dt^3} F(0) = \frac{d^3}{dt^3} F(T) = 0$$

Higher order differentials can be zeroed if required by extending the process outlined below. For example, a jerk free join requires that the fourth derivative is zero.

In the last section, a parametrization function to increase the momentum was constructed. The resulting motion primitive can be combined with other motion primitives which decreasing and maintain momentum to form a total smooth trajectory as shown in Figure 2.

A time dependent momentum function $\mathbf{p}(t) \Longrightarrow \mathbf{p}(F(\tau)) f(\tau)$ is considered. The rate of change of momentum is given by

$$\frac{d}{d\tau}(\mathbf{p}f) = f\frac{d}{d\tau}\mathbf{p} + \mathbf{p}\frac{d}{d\tau}f \tag{2.2}$$

The first term relates to the force needed to maintain the momentum at the level $\mathbf{p}f(\tau)$ and relates to the differential equations of motion for the system. The second term is the force needed to change the momentum as determined by the parametrization function. The second differential is

$$\frac{d^2}{d\tau^2}(\mathbf{p}f) = f\frac{d^2}{d\tau^2}\mathbf{p} + 2\frac{d}{d\tau}\mathbf{p}\frac{d}{d\tau}f + \mathbf{p}\frac{d^2}{d\tau^2}f \tag{2.3}$$

The first term arises from the moving frame of reference. The last term arises from the parametrization function only.

Referring to equations (2.2) and (2.3), for a smooth trajectory, it is also necessary for $\frac{d}{d\tau}\mathbf{p}(F(\tau))$ and $\frac{d^2}{d\tau^2}\mathbf{p}(F(\tau))$ to be smooth at the joins.

2.5. *Geometric Control Theory*

Geometric Control theory uses the expression $g^{-1}\frac{d}{dt}g(t) = \nabla H(t)$ to identify the optimal path in the configuration space of g from the identity to a configuration $g(T)$. (Bloch (2003) p336 provides a clear introduction). $\nabla H = \frac{dH}{dp}$ where H is the Hamiltonian optimized over the whole path and p is the momentum. The configuration space determines the interdependencies between the velocity components (cf the third part of equation (3.2)). The tangent field $\frac{d}{dt}g(t)$ and body frame are both pulled back to the identity by the action of g^{-1}. The initial momentum is dependent on the target configuration $g(T)$. Subsequently, the momentum (expressed in the body frame) is determined by the Hamiltonian and the configuration space of g.

Conditions on the velocity are not considered when optimizing the Hamiltonian. The initial momentum is modified by the parametrization function (possibly to zero). Subsequently, the optimal momentum is modified to achieve the target configuation and final momentum.

A parametrization function based on construction in the previous section changes the momentum (and the Hamiltonian) to achieve the greatest change in configuration given the constraints on the rate of change of momentum.

3. Asymmetric Rigid Body

An asymmetric rigid body is used to illustrate the motion planning process.

- The rotation is expressed in closed form based the three distinct inertia parameter $\{o_1, o_2, o_3\}$ defined in principle body frame, the kinetic energy H and the magnitude of the angular momentum k.
- The direction of the angular momentum is determined from the target attitude, and represented by the relationship between H and k.
- Finally, the profile of the angular momentum during the rotation is modeled by a time parametrization function.

3.1. *Momentum of freely rotating asymmetric body*

The rotation of a rigid body is usually expressed in the principle body frame. This is defined so that the moments of inertia about the 3 axes can be written as $\{o_1, o_2, o_3\}$. Cross terms only arise when the centre of mass is not at the origin of the frame.

The angular momentum of a freely rotating rigid body $\mathbf{k}$ is invariant (in magnitude and direction), and can be expressed as 3 orthogonal components $\{k_1, k_2, k_3\}$ so that

$$\mathbf{k} \equiv \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \text{ in } \mathfrak{so}(3) \text{ notation} \tag{3.1}$$

The constant magnitude $|\mathbf{k}|$, total kinetic energy H and the well known differential equations of motion gives rise to 3 equations

$$|\mathbf{k}|^2 = K = k_1^2 + k_2^2 + k_3^2, \quad H = \frac{k_1^2}{2o_1} + \frac{k_2^2}{2o_2} + \frac{k_3^2}{2o_3}, \quad \frac{d}{dt}k_i = \frac{o_j - o_k}{o_j o_k} k_j k_k \tag{3.2}$$

by permuting the indices $\{i, j, k\}$. From these 3 equations, it is possible to eliminate $k_1$and k_3 and obtain the equation

$$\left(\frac{d}{dt}k_2\right)^2 = \alpha k_2^4 + \beta k_2^2 + \gamma \tag{3.3}$$

where

$$\alpha = \frac{(o_1 - o_2)(o_2 - o_3)}{o_1 o_2^2 o_3}$$

$$\beta = \frac{4Ho_1o_3 - 2H(o_1 + o_3)o_2 + K(2o_2 - o_1 - o_3)}{o_1 o_2 o_3}$$

$$\gamma = \frac{2HK(o_1 + o_3) - K^2 - 4H^2 o_1 o_3}{o_1 o_3}$$

When the principle body frame is aligned so that $o_1 < o_2 < o_3$, $\alpha > 0$.

Equation (3.3) is rewritten into the form

$$\frac{1}{\gamma}\left(\frac{d}{dt}k_2\right)^2 = \left(1 - \frac{k_2^2}{a^2}\right)\left(1 - \frac{k_2^2}{b^2}\right)$$

where $\left\{a^2, b^2\right\} = \frac{-\beta \mp \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$.

The Jacobi elliptic functions are defined by $x = \mathrm{sn}\,(u, m) = \sin(\theta)$ where $\left(\frac{dx}{du}\right)^2 \equiv \left(1 - x^2\right)\left(1 - mx^2\right)$ and θ is known as the amplitude, $\theta = \mathrm{am}\,(u, m)$. The other basic Jacobi functions are $\mathrm{cn}\,(u, m) = \cos(\theta)$ and $\frac{d}{dt}\theta = \frac{d}{dt}u\,\mathrm{dn}\,(u, m)$.

The solution to equation (3.3) can be written as

$$k_2 = o_2\, a_2 \mathrm{sn}\ \left(wt + \phi, m\right)\ with\ o_2 a_2 = a,\ w = \frac{\sqrt{\gamma}}{a},\ m = \left(\frac{a}{b}\right)^2$$

(Refer to Marsden & Ratiu (1999) page 503, noting that the moments of inertia are sequenced differently)

If $2H = \frac{K}{o_2}$, then $K = k_2^2$ and all the rotation is about the second axis.

If $2H > \frac{K}{o_2}$, then $|k_1| > 0$ for all time and the parameter $m \in (0, 1)$. The components can be written to be

$$\mathbf{k} = \{o_1 a_1 \mathrm{dn}\ (wt + \phi, m), o_2 a_2 \mathrm{sn}\ (wt + \phi, m), o_3 a_3 \mathrm{cn}\ (wt + \phi, m)\} \tag{3.4}$$

If $2H < \frac{K}{o_2}$, then $|k_3| > 0$ for all time. The parameter $m \in (0, 1)$ and the components can be written to be $\{o_1 a_1 \mathrm{cn}\ (wt + \phi, m), o_2 a_2 \mathrm{sn}\ (wt + \phi, m), o_3 a_3 \mathrm{dn}\ (wt + \phi, m)\}$. In this case, the assumed expression for g in the next section would be written in an alternative form.

3.2. *Asymmetric satellite realignment*

One way that any rotation g can be defined is as $g = \exp(\varphi_3 \mathbf{e}_1) \exp(\varphi_2 \mathbf{e}_2) \exp(\varphi_1 \mathbf{e}_1)$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the basis of the tangent space. This is easily expanded to

$$g(t) = \begin{bmatrix} c_2 & s_1.s_2 & -c_1.s_2 \\ s_2.s_3 & c_1.c_3 - c_2.s_1.s_3 & -c_3.s_1 - c_1.c_2.s_3 \\ -c_3.s_2 & c_2.c_3.s_1 - c_1.s_3 & c_1.c_2.c_3 - s_1.s_3 \end{bmatrix} \text{ in } SO(3) \text{ notation} \tag{3.5}$$

where $c_1 = \cos(\varphi_1)$, $s_1 = \sin(\varphi_1)$ and similarly for the sin and cos of the angles φ_2 and φ_3.

The angular velocity found in the previous section is

$$\mathbf{w} = \{a_1 \mathrm{dn}\,(wt + \phi, m), a_2 \mathrm{sn}\,(wt + \phi, m), a_{23} \mathrm{cn}\,(wt + \phi, m)\} \tag{3.6}$$

Expressing this in the notation shown in equation (3.1), expanding the standard Lie group equation $\mathbf{w} = g^{-1}\frac{d}{dt}g$ and using three independent elements of the resultant matrix, we obtain

$$\frac{d}{dt}\varphi_1 + \frac{d}{dt}\varphi_3 \cos(\varphi_2) = a_1 \mathrm{dn}\,(wt + \phi, m) \tag{3.7}$$

$$\frac{d}{dt}\varphi_3 \sin(\varphi_2) = a_2 \mathrm{sn}\,(wt + \phi, m) \sin(\varphi_1) + a_3 \mathrm{cn}\,(wt + \phi, m) \cos(\varphi_1) \tag{3.8}$$

$$\frac{d}{dt}\varphi_2 = a_2 \mathrm{cn}\,(wt + \phi, m) \sin(\varphi_1) - a_3 \mathrm{sn}\,(wt + \phi, m) \cos(\varphi_1)$$

This has solution $\varphi_1 = \mathrm{am}\,(wt + \phi, m)$ so that $\frac{d}{dt}\varphi_1 = (a_1 - w)\,\mathrm{dn}\,(wt + \phi, m)$.

Introducing the parameter $l = \frac{d}{dt}\varphi_3$ and using equation (3.7), we can write

$$\cos(\varphi_2) = \frac{(a_1 - w)\,\mathrm{dn}\,(wt + \phi, m)}{l}$$

From equation (3.8),

$$\sin(\varphi_2) = \frac{w_{23}}{l}\ where\ w_{23} = a_2 \mathrm{sn}\,(wt + \phi, m)^2 + a_3 \mathrm{cn}\,(wt + \phi, m)^2 \tag{3.9}$$

Hence

$$l^2 = (a_1 - w)^2\, \mathrm{dn}\,(wt + \phi, m)^2 + \left(a_2 \mathrm{sn}\,(wt + \phi, m)^2 + a_3 \mathrm{cn}\,(wt + \phi, m)^2\right) \tag{3.10}$$

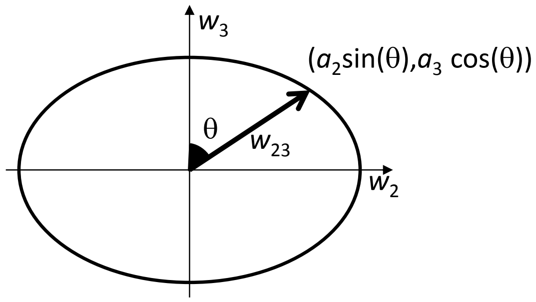
$$\varphi_3 = \int_0^t l dt \tag{3.11}$$

The rotation resulting from the angular velocity given in equation (3.6) is found by substituting these values into the general rotation in equation (3.5) and pulling it back to start at the identity using equation (3.12).

$$g^{-1}(0)\, g(t) \tag{3.12}$$

Based on the explanation in Marsden & Ratiu (1999), Figure 3 shows the second and third angular velocity components $(a_2 \sin(\theta), a_3 \cos(\theta))$ rotating about the first component. The angular velocity of precession is $\frac{d}{dt}\theta$ and the magnitude w_{23} was given in equation (3.9). In the asymmetric case,

$$\sin(\theta) \equiv \sin(\mathrm{am}\,(wt + \phi, m)) \equiv \mathrm{sn}\,(wt + \phi, m)$$



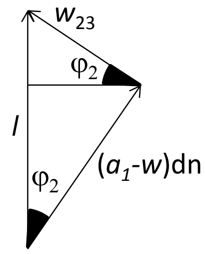


FIGURE 3. Determining the total angular velocity

and $\frac{d}{dt}\theta = w\mathrm{dn}\,(wt + \phi, m)$.

The total angular velocity l about the axis of rotation can be found analytically using equation (3.10) but the angle of rotation, equation (3.11), cannot be integrated analytically.

Note that l is the total angular velocity about the fixed axis of rotation and includes the angular velocity of precession, $\frac{d}{dt}\theta = w\mathrm{dn}\,(wt + \phi, m)$.

3.3. *Time dependent velocity*

Time parametrization of a time dependent velocity is illustrated for the asymmetric satellite. The invariant momentum is parametrized so that

$$\mathbf{k} \Longrightarrow f\mathbf{k}$$

With no external forces, there are two invariants: the angular momentum K and the energy H. The time parametrization function is applied to both these functions so that

$$K \Longrightarrow f^2K \, and \, H \Longrightarrow f^2H$$

Applying this parametrization to the equations in Section 3.1, the parameters $\{\alpha, \beta, \gamma\}$, $\{a, b\}$ and hence $\{w, m\}$ are changed as follows

$$\alpha \Longrightarrow \alpha,\ \beta \Longrightarrow f^2\beta,\ \gamma \Longrightarrow f^4\gamma$$

$$a \Longrightarrow fa,\ b \Longrightarrow fb$$

$$w \Longrightarrow fw,\ wt \Longrightarrow wF,\ m \Longrightarrow m$$

The resultant rotation is found by substituting into equation (3.5) with

$$s_1 = \sin(\varphi_1) \Longrightarrow \mathrm{sn}\,(wF + \phi, m)\,,\ c_1 = \cos(\varphi_1) \Longrightarrow \mathrm{cn}\,(wF + \phi, m)$$

$$s_2 = \sin(\varphi_2) \Longrightarrow \frac{f\left(a_2\mathrm{sn}\,(wF + \phi, m)^2 + a_3\mathrm{cn}\,(wF + \phi, m)^2\right)}{l_f},\ c_2 = \cos(\varphi_2) \Longrightarrow \frac{f\,(a_1 - w)\,\mathrm{dn}\,(wF + \phi, m)}{l_f}$$

with $l_f^2 = f^2\,(a_1 - w)^2\,\mathrm{dn}\,(wF + \phi, m)^2 + f^2\left(a_2\mathrm{sn}\,(wF + \phi, m)^2 + a_3\mathrm{cn}\,(wF + \phi, m)^2\right)^2$

Numeric methods are required to find $\varphi_{3,f} = \int_0^t l_f dt$ and substitute into

$$s_3 = \sin(\varphi_3) \Longrightarrow \sin(\varphi_{3,f})\,,\ c_3 = \cos(\varphi_3) \Longrightarrow \cos(\varphi_{3,f})$$

4. Satellite Example

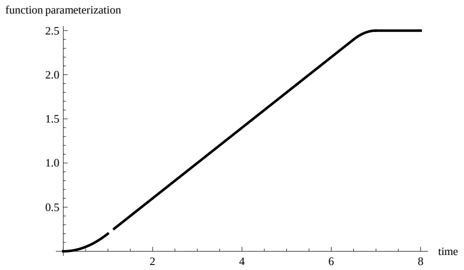
Throughout this example, the international system of units of measure (SI) is used. Velocity is kilometers/second. Moment of inertia is kilogram meters. Angular velocity is radians /second.

A satellite (with moments of inertia of $\{0.5, 1.0, 1.3\}$) is used as an example. The momentum is 1 and the energy is 0.6. The angular momentum about a fixed axis is found to be $\{b_1\mathrm{dn}(wt, m), b_2\mathrm{sn}(wt, m), b_3\mathrm{cn}(wt, m)\}$ in the principle body frame, with $\{b_1, b_2, b_3\} = \{0.945, 0.753, 0.325\}$, $w = 0.504$ and $m = 0.515$. It is increased from zero using a parametrization function derived as in Section 2.3 and illustrated on the left of Figure 4. The resultant angular velocity l_f is shown on the right, and reflects the tumbling of the principle body frame.

The linear motion is parametrized independently. The linear velocity $\{3, 0.5, 0\}$ as measured in the inertial frame is parametrized to $\{3, 0.5f, 0\}$ so that the centre of mass is at $\{3t, 0.5F\,(t)\,, 0\}$.

This is equivalent to anisotropically stretching the cotangent space. The linear momentum is stretched anisotropically in the inertia frame. In the body frame, the stretching varies with the rotation.

Figure 5 shows the resultant motion, consisting of an increasing angular momentum about a fixed axis of



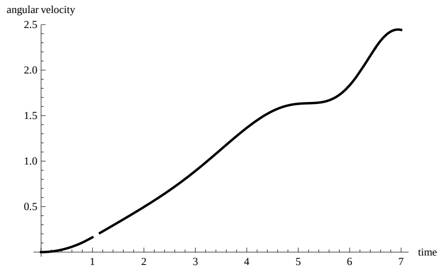


FIGURE 4. Parametrization function and Angular velocity

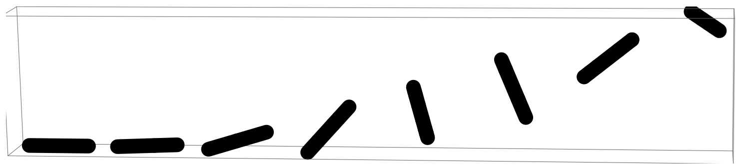


FIGURE 5. Accelerating satellite

rotation, a fixed linear momentum in the first direction and a sideways motion introduced in an orthogonal direction.

5. Conclusion

Time parametrization changes the momentum by isotropically stretching the cotangent space in which momentum is conserved in the absence of external forces. The original path is tracked with modified speed, enabling velocity boundary conditions to be imposed. This paper extends the parametrization from always starting and finishing at rest to starting and finishing at any multiple of the original momentum.

A parametrization function is identified which enables the longest path to be achieved in a fixed time, given the actuators' constraints on the maximum momentum, and its increasing and decreasing rates of change.

The optimal Hamiltonian of Geometric Control theory minimizes a cost function over the whole trajectory. A target configuration determines the initial momentum required. Parametrization enables the momentum to be modified over the trajectory to match constraints.

When the forces causing the change induced by the parametrization functions are removed, the vehicle continues to move along a path that minimizes the cost function.

The momentum can be modified by applying multiple parametrization functions, which is equivalent to anisotropic stretching of the momentum space. The momentum is modified to change direction as well as magnitude. For example, angular momentum can be introduced to turn a corner.

REFERENCES

AGUIAR, A. PEDRO, HESPANHA, JOÃO P., & KOKOTOVIĆ, PETAR V. 2008. Performance limitations in reference tracking and path following for nonlinear systems. *Automatica*, **44**(3), 598–610.

BIGGS, JAMES D., & HORRI, NADJIM M. 2012. Optimal geometric motion planning for spin-stabilized spacecraft. *System and control letters*.

BLOCH, ANTHONY M. 2003. *Nonholonomic Mechanics and Control: With the Collaboration of J.Baillieul, P.Crouch and J.Marsden (Interdisciplinary Applied Mathematics)*. Springer.

CHATURVEDI, NALIN, SANYAL, AMIT K., & MCCLAMROCH, N. HARRIS. 2011. Rigid-Body Attitude Control. *Ieee control systems*, **31**(3), 30–51.

FLASSKAMP, KATHRIN, & OBER-BLOBAUM, SINA. 2012. Energy efficient control for mechanical systems based on inherent dynamical structures. *Pages 2609–2614 of: 2012 american control conference*.

FRAZZOLI, EMILIO, DAHLEH, MUNTHER A, & FERON, ERIC. 2005. Maneuver-based motion planning for nonlinear systems with symmetries. *Ieee transactions on robotics*, **21**(6), 1077–1091.

LEE, GEON, KIM, JINHYUN, & CHOI, YOUNGJIN. 2012. Convolution-Based Trajectory Generation Methods Using Physical System Limits. *Journal of dynamic systems, measurement, and control*, **135**(1), 011001.

LOIZOU, SAVVAS G. 2012. Navigation Functions in Topologically Complex 3-D Workspaces. *Pages 4861–4866 of: 2012 american control conference*.

MACLEAN, CRAIG, PAGNOZZI, DANIELE, & BIGGS, JAMES D. 2011. Computationally light attitude controls for resource limited nano-spacecraft. *Pages 1–9 of: 62nd international astronautical congress 2011, cape town, south africa*.

MARSDEN, JERROLD E., & RATIU, TUDOR S. 1999. *Introduction to Mechanics and Symmetry*. Second edn. Springer.

PHAM, QUANG-CUONG. 2013. A General, Fast, and Robust Implementation of the Time-Optimal Path Parameterization Algorithm. *arxiv:1312.6533v2*, 9.

Smith, Ryan N. 2008. *Geometric Control Theory And Its Application To Underwater Vehicles.* Ph.D. thesis, Hawaii.

Spindler, Karlheinz. 1997. New Attitude Control algorithms with applications to Space telescopes. *Pages 227–232 of: 12th international symposium on 'space flight dynamics', darmstadt, germany.*

Spindler, Karlheinz. 2002. Motion Planning Via Optimal Control Theory. *In: Proceedings of the american control conference.*

Xargay, Enric, Kaminer, I, Pascoal, Antonio M., Hovakimyan, Naira, Dobrokhodov, Vladimir, Cichella, V., Aguiar, A. Pedro, & Ghabcheloo, R. 2013. Time-Critical Cooperative Path Following of Multiple Unmanned Aerial Vehicles over Time-Varying Networks. *Jornal of guidance, control and dynamics*, **36**(2), 499–516.

Yang, Gil Jin, & Choi, Byoung Wook. 2013. Smooth Trajectory Planning Along Bezier Curve for Mobile Robots with Velocity Constraints. *International journal of control and automation*, **6**(2), 225–234.