IMA Conference on Mathematics of Robotics
9 – 11 September 2015, St Anne's College, University of Oxford

Kinematic Singularities of Mechanisms Revisited

By Andreas Müller[1], Dimiter Zlatanov[2]
[1]Johannes Kepler University, Linz, Austria; [2]University of Genoa, Genoa, Italy

Abstract

The paper revisits the definition and the identification of the singularities of kinematic chains and mechanisms. The degeneracy of the kinematics of an articulated system of rigid bodies cannot always be identified with the singularities of the configuration space. Local analysis can help identify kinematic chain singularities and better understand the way the motion characteristics change at such configurations. An example is shown that exhibits a kinematic singularity although its configuration space is a smooth manifold.

1. Introduction

Kinematic singularities of a mechanism are critical configurations that can lead to a loss of structural stability or controllability. This has been a central topic in mechanism theory and still is a field of active research.

A systematic approach to the study of singular configurations involves a mathematical model for the kinematic chain and its interaction with the environment via inputs and outputs. Thereupon critical configurations can be identified for the kinematic chain itself and for the input and output relations.

A *kinematic chain* is a system of rigid bodies (*links*), some pairs of which are connected with *joints*. It is defined by specifying exactly which links are jointed (by a connectivity graph [Wittenburg (1994)]), the type of each joint, and the joint's locations in the adjacent links. Mathematically, a kinematic chain is modeled by specifying its possible motions as a subset of the smooth curves on an ambient manifold, usually assumed to have a global parametrization, $\mathbb{V}^n$. There may be different sensible ways to choose $\mathbb{V}^n$. The two most important approaches involve the parametrization of the link and the joint space, respectively, sometimes via embedding in a higher-dimensional space). The feasible configurations form the configuration space (c-space) $V \subset \mathbb{V}^n$ defined by geometric constraint equations. It will be assumed that these can be written in terms of the global coordinates of $\mathbb{V}^n$ only, i.e., the constraints are holonomic and scleronomic (do not include explicitly either time or any derivatives of the coordinate functions).

A mechanism is a kinematic chain used to transmit motion. This is understood in the sense that a mechanism is intended to be an input-output device. Mathematically, this is modeled by introducing an input and an output space. Typically, for a manipulator, an output space would be an $SE(3)$ submanifold containing and of the same dimension as the workspace of the end-effector, while the input space would be a Cartesian product of the actuated joint spaces. Each motion of the kinematic chain defines unique input and output motions in these spaces, via the input and the output maps, defined on the ambient space $\mathbb{V}^n$ [Zlatanov (1998)]. It is important to note that the input space is the target, not the source, of the input map. In this sense the input mapping defines a (local) chart on the c-space.

In a regular configuration, a non-redundant mechanism allows to control the motion in the output space by choosing a motion in the input space, and vice versa. The theory and the analysis of mechanism singularities studies the cases where this is not possible. In particular, degenerate or singular configurations are usually defined as those where either the input, or the output map cannot be inverted. When this happens one can distinguish two most general cases:

(a) The lack of invertibility is due to some degeneracy of the local properties of the *kinematic chain*; no choice of input or output would make the resulting mechanism function regularly at this configuration. The singular phenomenon can be identified by analyzing the constraints.

(b) The constraints of the kinematic chain do not degenerate at the configurations. The lack of invertibility is specific to the input or output maps.

In this paper, the case (a) will be considered. Often this type of phenomena is referred to as c-space singularity. The configuration space, V, is the set of all feasible configurations (the points in $\mathbb{V}^n$ which satisfy the constraints). Selig (2005) credits Gibson for the introduction of this key notion in mechanism theory. It would seem that for holonomic and scleronomic constraints the geometry of the configuration space completely describes the kinematic chain. However, as we discuss below, this is not always so. The reason lies in the fact that the mobility of a kinematic chain is essentially defined by the set of feasible motions rather than a the set of feasible configurations. Moreover, for both practical and theoretical reasons, this must include *instantaneous motions*. The feasible instantaneous motions can be determined from the constraints, but not necessarily from the configuration space V.

In this paper, the notion of kinematic-chain singularity is reviewed. A definition is proposed that includes established concepts. C-space singularities and constraint singularities are distinguished, and are related to 'kinematic singularities' of the kinematic chain. The kinematic chain and its constraints are modeled in terms of joint angles and displacements using the product of exponentials. Then, the configuration space, V, is an analytic variety. An approach to the local analysis of this is presented that aids identification of singularities. This is demonstrated for two examples.

2. Constraints, Configuration Space, and Mobility

The kinematics of an articulated system of rigid bodies is determined by the constraints its members are subjected to. A system of m geometric constraints $h(\mathbf{q}) = \mathbf{0}$ with constraint mapping

$$h : \mathbb{V}^n \to \mathbb{R}^m \tag{2.1}$$

determines the *configuration space* (c-space)

$$V := h^{-1}(\mathbf{0}). \tag{2.2}$$

This is a variety in the parameter space $\mathbb{V}^n$. Depending on the parametrization used and the nature of the constraints, V is an algebraic or an analytic variety.

The geometric constraints give rise to a system of m velocity constraints of the form

$$\mathbf{J}(\mathbf{q})\,\dot{\mathbf{q}} = \mathbf{0} \tag{2.3}$$

that determine the admissible first-order motions as $K_{\mathbf{q}} := \ker \mathbf{J}(\mathbf{q})$, where $\mathbf{J}$ is the Jacobian of h. The *differential* (or instantaneous) DOF of a kinematic chain at $\mathbf{q} \in V$ is $\delta_{\text{diff}}(\mathbf{q}) := \dim K_{\mathbf{q}} = n - \text{rank}\,\mathbf{J}(\mathbf{q})$, which is intrinsically determined by the constraints rather then the geometry of V.

IMA Conference on Mathematics of Robotics
9 – 11 September 2015, St Anne's College, University of Oxford

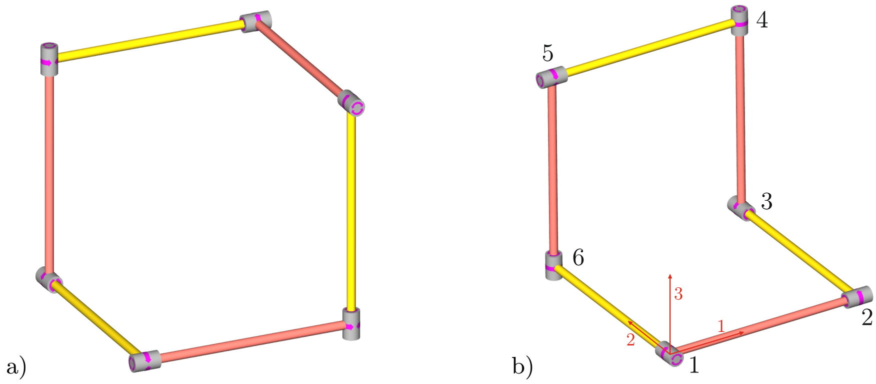


FIGURE 1. Two assembly modes of a 6R chain. a) Immobile 6R linkage. b) Line-symmetric Bricard linkage. The c-space V consists of an isolated point and a smooth curve.

The dimension of the c-space variety reveals the finite mobility and DOF of the kinematic chain, which is thus a geometric property of the c-space. The *local DOF* at a given configuration is the local dimension of V, denoted $\delta_{\text{loc}}(\mathbf{q}) := \dim_{\mathbf{q}} V$. The latter is the highest dimension of manifolds in V passing through $\mathbf{q}$. The *global DOF* of a linkage is the highest local DOF. If V consists of disconnected components, i.e. the linkage possesses different assembly modes, the global DOF should be restricted to the assembly mode of interest. For example the two assembly modes of the 6R linkage in fig. 1 correspond to two disconnected subvarieties of V. The assembly in fig. 1a) is well-constrained, i.e. immobile [Chai & Chen (2010)], corresponding to a 0-dim submanifold. The 6R linkage can also be assembled to the line-symmetric Bricard mechanism in fig. 1b) possessing 1 DOF. The c-space of the latter is a 1-dim smooth submanifold. It is clearly impossible to change from one to the other assembly mode without disassembling the linkage, i.e. V consists of two disconnected components, one is a single point and the other a smooth curve. If a connected subvariety of V comprises submanifolds with different dimensions, i.e. different local DOF, the linkage is said to be *kinematotropic* [Wohlhart (1996)]. Such linkages can hence change their finite DOF by a motion from one to the other manifold, resembling what is recently referred to as reconfigurability. This transition is accompanied by a change of the differential mobility. This does not necessarily mean that the motion is a non-smooth curve in V. A noticeable property of special linkages is that they exhibit smooth kinematotropies, i.e. the corresponding motion curve is smooth, which is deemed to have practical significance as it avoids motion discontinuities [Müller & Piipponen (2015)]. The general conditions on the linkage geometry are yet unknown.

It is important to stress that for special geometries, the differential mobility is not always determined by the c-space. The 7R linkage in the configuration $\mathbf{q}_1$ shown in fig. 2b) possesses $\delta_{\text{loc}}(\mathbf{q}_1) = 1$ while $\delta_{\text{diff}}(\mathbf{q}_1) = 3$. Yet the c-space is locally a smooth curve. Moreover, this is not an artifact due to the parameterization with joint angles but inherent to the linkage. Hence, $K_{\mathbf{q}} \neq T_{\mathbf{q}} V$, and the differential geometry of V does not reveal the instantaneous mobility. The configuration in fig. 2a) where the linkage can bifurcate to different motion modes is indeed reflected in the c-space.

3. Singularities

The motion of a kinematic chain is a curve in its c-space. Its finite mobility is encoded in the c-space geometry, whereas the instantaneous mobility cannot necessarily be deduced from the geometry of V. The interest here are 'critical configurations' or 'singularities'

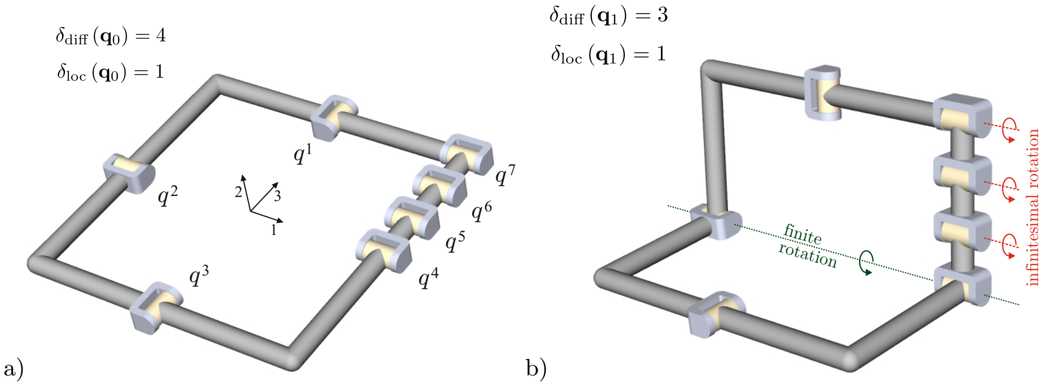


FIGURE 2. a) Singular configuration $\mathbf{q}_0$ of a 7R linkage with special geometry. b) Regular configuration $\mathbf{q}_1$. Still the local and differential DOF are different.

of the kinematic-chain. At such a configuration the motion capabilities of the articulated system undergo a qualitative change. This basic characteristic of interest is the mobility or DOF. The finite DOF is a local property whereas the differential DOF is a pointwise property. Therefore the latter indicates kinematic singularities. This may or may not be reflected in the c-space.

3.1. *Constraint Singularities*

A *constraint singularity* of a kinematic chain is a configuration $\mathbf{q} \in V$ which is a critical point of the constraint mapping, i.e. the constraint Jacobian is not full rank. Constraint singularities are not necessarily critical configurations of the kinematic chain. Moreover it is well-known that overconstrained linkages possess a manifold of regular points with singular Jacobian. The Bricard linkage in fig. 1b) is prominent example. Its c-space is a 1-dim smooth manifold on which the Jacobian is singular with corank 1.

3.2. *Kinematic Singularities of the Linkage*

The configuration $\mathbf{q} \in V$ is a *kinematic singularity* of the kinematic chain iff the differential DOF δ_{diff} is not constant in any neighborhood of $\mathbf{q}$ in V. Otherwise the configuration is *regular*. At a kinematic singularity, the constraint Jacobian is necessarily not full rank m. The reverse is not true, however, since regular points of an overconstrained kinematic chain are constraint singularities.

3.3. *C-Space Singularities*

How is a kinematic singularity reflected in the c-space? A configuration $\mathbf{q}$ is a *singular point of the c-space*, iff V is not a smooth manifold at $\mathbf{q}$. Clearly at a c-space singularity the differential DOF changes, so that a c-space singularity is always a kinematic singularity as well as a constraint singularity. However, the reverse is not true, since there are situations where V is smooth but the differential DOF is not constant. This is phenomenon seems to be relatively unknown.

As example consider the Goldberg 6R linkage, [Feng et al. (2015)], in fig. 3 with geometry described by the DH parameters $\alpha_{12} = 3/4\pi, \alpha_{23} = \alpha_{61} = \pi/4, \alpha_{34} = \pi/2, \alpha_{45} = -\pi/4, \alpha_{56} = \pi/2, a_{12} = a_{34} = a_{56} = 1, a_{23} = a_{61} = 1/\sqrt{2}, a_{45} = -1/\sqrt{2}, R_i = 0, i = 1, \ldots, 6$. It is mobile with $\delta_{\text{loc}} = 1$. Also its differential DOF is $\delta_{\text{diff}} = 1$ except at the 'stretched' configuration $\mathbf{q}_0$ shown in fig. 4 where $\delta_{\text{diff}}(\mathbf{q}_0) = 2$. Contrary to what might be expected, this is not a c-space singularity, and the c-space is a smooth curve. Yet the linkage exhibits kinematic singularities (see analysis in sec. 4).

IMA Conference on Mathematics of Robotics
9 – 11 September 2015, St Anne's College, University of Oxford

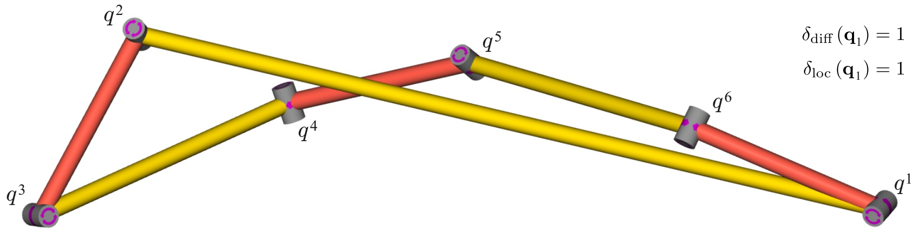


FIGURE 3. 6R Goldberg linkage in a regular configuration.

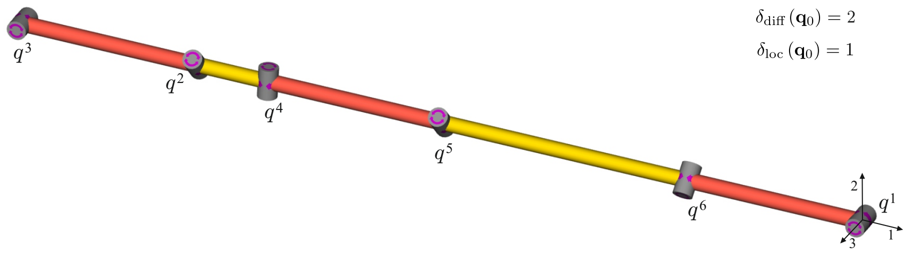


FIGURE 4. 6R Goldberg linkage in a kinematic singularity without bifurcation. This type was classified as an IIM-type singularity.

This type of phenomenon is known to occur for algebraic varieties, but to the best of our knowledge it has not been reported for kinematic chains. For example the real planar curve defined by the equation $y^3 + 2yx^2 - x^4 = 0$ defines an analytic manifold, but the point $(0, 0)$ is singular. The reason is that complex branches intersect the smooth real branch at the origin, Milnor (1969).

4. Singularity Analysis of Linkages

4.1. *Constraints*

The motions of a kinematic chain occur in its c-space. The c-space itself can be defined by various equations, and the system of constraints of a chain with specific geometry is only one possibility. In other words, the constraints are merely a particular set of generators of the variety V. Yet, they define the chain's motion. Hence, the detection and analysis of kinematic singularities requires investigation of the specific constraints together with the c-space. The latter requires local higher-order or global considerations.

Constraints for kinematic chains with so-called algebraic joints can be formulated as a system of polynomials. In this algebraic setting the central object is the polynomial ideal of V, allowing for an analysis of the c-space geometry regardless of the particular constraints by means of computational geometry algorithms. The challenge for this approach is that it leads to NP hard problems.

Alternatively, the kinematics of linkages, i.e., kinematic chains whose joints can be modeled as combination of lower pairs, can be described by successive screw motions. The closure constraints for kinematic loop comprising n lower kinematic pairs are then

$$f(\mathbf{q}) = \mathbf{I} \tag{4.1}$$

with the constraint mapping $f : \mathbb{V}^n \to SE(3)$ given by the product of exponentials (POE) formula $f(\mathbf{q}) = \exp(\mathbf{Y}_1 q^1) \exp(\mathbf{Y}_2 q^2) \cdot \ldots \cdot \exp(\mathbf{Y}_n q^n)$ [Brockett (1984), Selig (2005)]. Here $\mathbf{Y}_i = (\mathbf{e}_i, \mathbf{s}_i \times \mathbf{e}_i + h_i \mathbf{e}_i)$ is the screw coordinate vector of joint i w.r.t. to a global reference frame in the zero reference configuration $\mathbf{q} = \mathbf{0}$. The c-space of the kinematic loop is then the analytic variety $V := f^{-1}(\mathbf{I})$. This applies to multi-loop linkages by introducing constraints for each topologically independent fundamental cycle.

The right-trivialized derivative of f at $\mathbf{q} \in V$ yields the velocity constraints

$$\mathsf{S}_n^1(\mathbf{q}, \dot{\mathbf{q}}) := \sum_{i \leq n} \mathbf{S}_i(\mathbf{q})\, \dot{q}^i = \mathbf{0} \tag{4.2}$$

where $\mathbf{S}_i$ is the instantaneous screw coordinate vector of joint i. The latter is obtained by transforming the screw coordinates $\mathbf{Y}_i$ to the current configuration

$$\mathbf{S}_i = \mathbf{Ad}_{g_i} \mathbf{Y}_i \tag{4.3}$$

with $g_i(\mathbf{q}) = \exp(\mathbf{Y}_1 q^1) \cdot \ldots \cdot \exp(\mathbf{Y}_i q^i)$. The constraints can be written as (2.3) collecting the $\mathbf{S}_i$ in the *geometric Jacobian* $\mathbf{J}$.

Time derivatives of the velocity constraints are readily found with the following relation $\frac{\partial}{\partial q^j} \mathbf{S}_i = [\mathbf{S}_j, \mathbf{S}_i], j < i$. Moreover repeated partial derivatives are given by nested Lie brackets (screw products) involving simple cross products [Müller (2014)]. The time derivatives of the velocity constraints are thus known explicitly in terms of algebraic operations.

4.2. *Local Analysis*

A local analysis of V at $\mathbf{q}$ aims at locally approximating its geometry. The best local approximation of V at $\mathbf{q}$ is its tangent cone denoted $C_{\mathbf{q}}V$. This is the set of all tangent vectors to curves in V through $\mathbf{q}$ [Whitney (1965)]. That is, $C_{\mathbf{q}}V$ restricts vectors in $T_{\mathbf{q}}V$ to those that are tangent to curves through $\mathbf{q}$. Conversely, the tangent cone determines the tangent space as

$$T_{\mathbf{q}}V = \operatorname{span} C_{\mathbf{q}}V. \tag{4.4}$$

The tangent cone reveals possible finite motions through $\mathbf{q}$. Its dimension is the local DOF: $\dim C_{\mathbf{q}}V = \dim_{\mathbf{q}} V = \delta_{\text{loc}}(\mathbf{q})$. In general $C_{\mathbf{q}}V$ is a cone in $\mathbb{R}^n$. It is identical to the tangent vector space $T_{\mathbf{q}}V$ iff V is a smooth manifold at $\mathbf{q}$. The strict definition of tangent space and cone resorts to the ideal $\mathbf{I}(V, \mathbf{q})$ of analytic germs over $\mathbf{q}$, that vanish on V, Whitney (1965), just like their equivalents in algebraic geometry. This ideal is unknown, however.

Lerbet first applied the concept of tangent cone to the kinematic analysis of linkages [Lerbet (1999)]. The cone is introduced noticing its interpretation as set of tangents to curves. Any tangent $\dot{\mathbf{q}} \in \mathbb{R}^n$ must satisfy the ith-order constraints

$$H^{(i)}(\mathbf{q}; \dot{\mathbf{q}}, \ldots, \mathbf{q}^{(i)}) = \mathbf{0} \tag{4.5}$$

where the derivative

$$\begin{aligned} H^{(1)}(\mathbf{q}; \dot{\mathbf{q}}) &:= \mathsf{S}_n^1(\mathbf{q}, \dot{\mathbf{q}}) \\ H^{(2)}(\mathbf{q}; \dot{\mathbf{q}}, \ddot{\mathbf{q}}) &:= \frac{d}{dt} \mathsf{S}_n^1(\mathbf{q}, \dot{\mathbf{q}}) \\ &\cdots \\ H^{(i)}(\mathbf{q}; \dot{\mathbf{q}}, \ldots, \mathbf{q}^{(i)}) &:= \frac{d^{i-1}}{dt^{i-1}} \mathsf{S}_n^1(\mathbf{q}, \dot{\mathbf{q}}) \end{aligned} \tag{4.6}$$

are given in terms of nested Lie brackets.

Those $\dot{\mathbf{q}}$ (not necessarily tangent to V) that satisfy the constraints up to order i constitute the set

$$K^i_{\mathbf{q}} := \{\mathbf{x}|\exists \mathbf{y},\mathbf{z},\ldots \in \mathbb{R}^n : \begin{array}{l} H^{(1)}(\mathbf{q};\mathbf{x}) = \mathbf{0}, \\ H^{(2)}(\mathbf{q};\mathbf{x},\mathbf{y}) = \mathbf{0}, \\ H^{(3)}(\mathbf{q};\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{0}, \\ \cdots \\ H^{(i)}(\mathbf{q};\mathbf{x},\mathbf{y},\mathbf{z},\ldots) = \mathbf{0} \quad \} \end{array} \tag{4.7}$$

with $K^1_{\mathbf{q}} = \ker \mathbf{J}(\mathbf{q})$. The tangent cone is thus defined as limit of the inclusion

$$C_{\mathbf{q}}V = K^{\kappa}_{\mathbf{q}} \subset \ldots \subset K^3_{\mathbf{q}} \subset K^2_{\mathbf{q}} \subset K^1_{\mathbf{q}} \tag{4.8}$$

so that $K^{\kappa}_{\mathbf{q}} = K^{\kappa+1}_{\mathbf{q}}$. Tangent vectors to finite curves belong to $C_{\mathbf{q}}V$ and thus to all $K^i_{\mathbf{q}}$. The ith-order DOF at $\mathbf{q}$ is $\dim K^i_{\mathbf{q}}$. This is an infinitesimal DOF, iff $K^i_{\mathbf{q}} \neq C_{\mathbf{q}}V$. The tangent cone is hence the algebraic variety defined by the polynomials $H^{(i)}(\mathbf{q};\cdot)\,, i = 1,\ldots,\kappa$. It can be shown that each $K^i_{\mathbf{q}}$ is a cone, i.e. if $\mathbf{x} \in K^i_{\mathbf{q}}$, then also $\alpha\mathbf{x} \in K^i_{\mathbf{q}}$.

The output of the local analysis are the tangent space and the tangent cone, and thus the local DOF and mobility. This allows to identify c-space singularities.

In the following two examples are reported where the local analysis does not provide sufficient information to conclude whether a configuration is a kinematic singularity.

Example 1 (Goldberg 6R linkage). *The configuration $\mathbf{q}_0$ of the 6R linkage in fig. 3b) is analyzed. In the reference configuration $\mathbf{q}_0$, the joint screw coordinates are*

$$\begin{array}{ll} \mathbf{Y}_1 = (0,0,1,0,0,0)^T, & \mathbf{Y}_2 = (0,1/\sqrt{2},-1/\sqrt{2},0,1/2-\sqrt{2},1/2-\sqrt{2})^T \\ \mathbf{Y}_3 = (0,0,-1,0,-2,0)^T, & \mathbf{Y}_4 = (0,1,0,0,0,-1)^T \\ \multicolumn{2}{l}{\mathbf{Y}_5 = (0,1/\sqrt{2},-1/\sqrt{2},0,-1/2-1/\sqrt{2},-1/2-1/\sqrt{2})^T} \\ \multicolumn{2}{l}{\mathbf{Y}_6 = (0,1/\sqrt{2},1/\sqrt{2},0,1/2,-1/2)^T} \end{array}$$

The solution of the constraints (4.2) determines the first-order cone

$$K^1_{\mathbf{q}_0} = \{\mathbf{x} \in \mathbb{R}^6 | \mathbf{x} = (1/2((1-\sqrt{2})t-s), t-(1+\sqrt{2})s, 1/2(s+(1-\sqrt{2})t), s-\sqrt{2}t, s, t),\ s,t \in \mathbb{R}\}.$$

This is a 2-dim vector space, so that $\delta_{diff}(\mathbf{q}_0) = \dim K^1_{\mathbf{q}_0} = 2$. The second-order cone is

$$K^2_{\mathbf{q}_0} = \{\mathbf{x} \in \mathbb{R}^6 | \mathbf{x} = (-s,0,0,-(1+\sqrt{2})s, s, (1+\sqrt{2})s),\ s \in \mathbb{R}\}.$$

All higher-order cones are $K^i_{\mathbf{q}_0} = K^2_{\mathbf{q}_0}, i \geq 2$, so that $C_{\mathbf{q}_0}V = K^2_{\mathbf{q}_0}$, and $\delta_{loc}(\mathbf{q}_0) = \dim_{\mathbf{q}_0} V = 1$. Since $C_{\mathbf{q}_0}V$ is a 1-dim vector space (the tangent space to V) the c-space is a 1-dim smooth manifold at $\mathbf{q}_0$. Hence, $\mathbf{q}_0$ is a constraint singularity but not a c-space singularity. Whether it is a kinematic singularity cannot be decided. Even though $\delta_{loc}(\mathbf{q}_0) \neq \delta_{diff}(\mathbf{q}_0)$ it may be a regular configuration (see next example). In fact, it follows by inspection that δ_{diff} is not locally constant, and thus $\mathbf{q}_0$ a kinematic singularity.

Example 2 (7R linkage). *The joint screw coordinates of the 7R linkage in the configuration $\mathbf{q}_0$ in fig. 2a) w.r.t. the shown reference frame are*

$$\begin{array}{l} \mathbf{Y}_1 = \mathbf{Y}_3 = (0,0,1,0,0,0)^T,\ \mathbf{Y}_2 = \mathbf{Y}_4 = (1,0,0,0,0,0)^T \\ \mathbf{Y}_5 = (1,0,0,0,1/3,0)^T,\ \mathbf{Y}_6 = (1,0,0,0,2/3,0)^T,\ \mathbf{Y}_7 = (1,0,0,0,1,0)^T. \end{array}$$

Therewith the first-order cone

$$K^1_{\mathbf{q}_0} = \{\mathbf{x} \in \mathbb{R}^7 | \mathbf{x} = (-s,-t+u+2v,s,t,-2u-3v,u,v),\ s,t,u,v \in \mathbb{R})$$

IMA Conference on Mathematics of Robotics
9 – 11 September 2015, St Anne's College, University of Oxford

is 4-dimensional and $\delta_{diff}(\mathbf{q}_0) = 2$. *The second-order cone is* $K^2_{\mathbf{q}_0} = K^{2,1}_{\mathbf{q}_0} \cup K^{2,2}_{\mathbf{q}_0}$, *with*

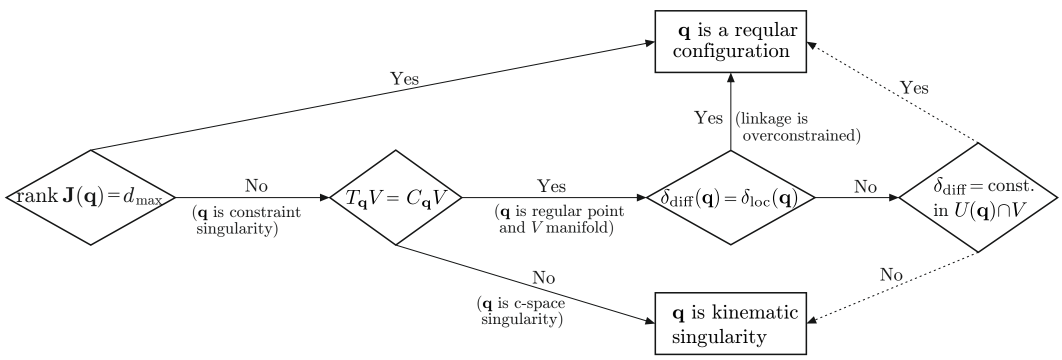
$$K^{2,1}_{\mathbf{q}_0} = \{\mathbf{x} \in \mathbb{R}^7 | \mathbf{x} = (0, s, 0, -s, 0, 0, 0),\ s \in \mathbb{R})$$
$$K^{2,1}_{\mathbf{q}_0} = \{\mathbf{x} \in \mathbb{R}^7 | \mathbf{x} = (s, 0, -s, 0, 0, 0, 0),\ s \in \mathbb{R}).$$

The tangent cone is $C_{\mathbf{q}_0}V = K^2_{\mathbf{q}_0}$, *which is thus the union of the two 1-dim vector spaces* $K^{2,1}_{\mathbf{q}_0}$ *and* $K^{2,2}_{\mathbf{q}_0}$ *that are the tangent spaces to the two manifolds* $V^1 = \{\mathbf{q} \in \mathbb{V}^7 | \mathbf{q} = (0, s, 0, -s, 0, 0, 0),\ s \in \mathbb{R})$ *and* $V^2 = \{\mathbf{q} \in \mathbb{V}^7 | \mathbf{q} = (s, 0, -s, 0, 0, 0, 0),\ s \in \mathbb{R})$ *intersecting at* $\mathbf{q}_0$. *The tangent space* $T_{\mathbf{q}_0}V = \mathrm{span}\ C_{\mathbf{q}_0}V \neq C_{\mathbf{q}_0}V$ *is a 2-dim vector space. Consequently,* $\mathbf{q}_0$ *is a c-space and a kinematic singularity, and thus a constraint singularity. A configuration* $\mathbf{q}_1 \in V^1$ *is shown in fig. 2b). The analysis shows that* $C_{\mathbf{q}_1}V = K^2_{\mathbf{q}_1}$ *is a 1-dim vector space. It is thus the tangent space* $T_{\mathbf{q}_1}V$ *to* V^1. *Hence,* $\mathbf{q}_1$ *is a regular point of* V, *i.e.* V *is locally a smooth manifold, and* $\delta_{loc}(\mathbf{q}_1) = \dim C_{\mathbf{q}_1}V = 1$. *Since* $\delta_{diff}(\mathbf{q}_1) = 3 > \delta_{loc}(\mathbf{q}_1)$, $\mathbf{q}_1$ *is a constraint singularity. It is not a kinematic singularity since* $\delta_{diff}(\mathbf{q}_1)$ *is locally constant. The latter statement follows from observation –it cannot be concluded from the analysis of* V*!*

Remark 1 ('Singular Linkages'). *Linkages, that possess permanently higher instantaneous than finite DOF, as in the last example, are certainly very special cases. These were termed 'shaky' [Wohlhart (1999)]. Frequently they are merely considered as 'singular'. This notion is rooted in the intuitive understanding that such linkages are indeed somehow special. In fact, such linkages are non-generic due to conditions imposed on the geometry. Considered as member of the family of mechanisms with the same kinematic topology, i.e. within the variety of linkages parameterized in terms of geometric parameters, and such linkages represent indeed a singularity –but only within that.*

5. Identification of Singularities

Constraint singularities are commonly detected by analyzing the rank of the constraint Jacobian, i.e. the instantaneous joint screw system. Strictly, this only indicates constraint singularities. The basic condition to be checked for $\mathbf{q}$ being a kinematic singularity is that $\delta_{\mathrm{diff}}(\mathbf{q}) \neq \delta_{\mathrm{diff}}(\mathbf{p}), \mathbf{p} \in U(\mathbf{q}) \cap V$ for any neighborhood $U(\mathbf{q})$. The above higher-order analysis provides a means for a partial identification. This is summarized in the following decision flow chart below. As indicated by the dashed lined, the situation where simultaneously $T_{\mathbf{q}_0}V = C_{\mathbf{q}_0}V$ and $\delta_{\mathrm{diff}} \neq \delta_{\mathrm{loc}}$ cannot be decided with the above method, and the dashed line merely means that some other means are to be invoked (as in the above examples).



6. Conclusions

The kinematics of a mechanism degenerates either because there is a kinematic-chain singularity or because the input or output map is at a critical point. The critical configurations of the kinematic chain, due only to the nature of the constraints and not to the specific input/output maps, should be distinguished from two related phenomena: singularities of the configuration space variety and the presence of redundant constraint equations. Of particular interest is the recently discovered phenomenon of linkages that exhibit kinematic singularities which are regular points of the c-space. Future work should focus on methods for the identification of non-bifurcating singularities and on the exact implications of such degeneracies on the input-output relationships of a mechanism

Acknowledgements

The authors of this paper would like to thank the authors of [Feng et al. (2015)], and Huijuan Feng in particular, for discussing the singularity of Goldberg 6R linkage. The first author acknowledges partial supported by the Austrian COMET-K2 program of the Linz Center of Mechatronics (LCM).

REFERENCES

R. W. Brockett: Robotic manipulators and the product of exponentials formula, Mathematical Theory of Networks and Systems, Lecture Notes in Control and Information Sciences Vol. 58, 1984, pp 120-129

W.H. Chai, Y. Chen: The line-symmetric octahedral Bricard linkage and its structural closure, Mech. Mach. Theory, 45 (2010) 772–779

J. Lerbet: Analytic Geometry and Singularities of Mechanisms, ZAMM, Z. angew. Math. Mech., Vol. 78, No. 10b, 1999, pp. 687-694

A. Müller: Higher Derivatives of the Kinematic Mapping and some Applications, Mechanism and Machine Theory, Vol. 76, 2014, pp. 70-85

A. Müller, S. Piipponen: On Regular Kinematotropies, 14th World Congress in Mechanism and Machine Science, Taipei, Taiwan, 25-30 October, 2015, submitted

J. Selig: Geometric Fundamentals of Robotics (Monographs in Computer Science Series), Springer-Verlag New York, 2005

J. Milnor: Singular Points of Complex Hypersurfaces (Annals of Mathematics Studies 61), Princeton University Press, 1968

A. H. Feng, R. Kang, Y. Chen, Jian S Dai: Kinematic Singularity and Bifurcation of Bennett-Based Overconstrained Linkages, submitted to the Journal of Mechanical Engineering Science, maniscript ID JMES-14-141814th, 2015

H. Whitney: Local properties of analytic varieties, Differential and Combinatorial Topology, Princeton University Press, 1965

K. Wohlhart: Kinematotropic Linkages, in: J. Lenarčič, V. Parent-Castelli (eds.): Recent Advances in Robot Kinematics, Kluwer, 1996, pp. 359-368

K. Wohlhart: Degrees of Shakiness, Mech. Mach. Theory, Vol. 34, 1999, pp. 1103-1126

J. Wittenburg: Topological description of articulated systems, in Pereira and Ambrosio (eds), Computer Aided Analysis of Rigid and Flexible Mechanical Systems, 1994, pp 159-196

D. Zlatanov: Generalized singularity analysis of mechanisms, Ph. D. thesis, University of Toronto, 1998

Zlatanov, D., Bonev, I.A., Gosselin, C.M.: Constraint Singularities as C-Space Singularities, 8th International Symposium on Advances in Robot Kinematics (ARK 2002), Caldes de Malavella, Spain, June 24-28, 2002