IMA Conference on Mathematics of Robotics
9 – 11 September 2015, St Anne's College, University of Oxford

Lagrangian Jacobian motion planning†

By Krzysztof Tchoń and Ida Góral

Chair of Cybernetics and Robotics, Electronics Faculty, Wrocław University of Technology, Wrocław, Poland

Abstract

This paper is devoted to the motion planning problem of nonholonomic robotic systems whose kinematics are represented by a driftless control system with outputs. The motion planning problem is defined in terms of the inversion of the end-point map of the system, that may be accomplished by a Jacobian algorithm. The Lagrangian Jacobian inverse is introduced and examined, and a corresponding Lagrangian Jacobian motion planning algorithm is designed. Performance of this algorithm is illustrated by solving a motion planning problem for the rolling ball.

1. Introduction

It is well known that the kinematics of a nonholonomic robotic system satisfying the Pfaffian motion constraints can be represented by a driftless control system. In order to describe the objective of motion planning, this system is equipped with an output function. In this setting the motion planning problem means finding a control of the system such that the output at a prescribed time instant reaches a desired point. Stated in this way, the motion planning problem can be reduced to the inversion of the end-point map of the control system, and solved using Jacobian inversion algorithms. This approach in mobile robotics, often called the continuation method approach, was initiated by Sussmann (1992), then developed by several authors, see Chitour & Sussmann (1992), Divelbiss, Seereeram & Wen (1998), Chitour (2006), and also extended to mobile manipulators under the name of the endogenous configuration space approach by Tchoń & Jakubiak (2003). In the literature most often the Jacobian pseudoinverse has been applied, coming from the solution of a constrained energy minimization problem in the linear approximation of the control system representation of the kinematics.

As a generalization of the Jacobian pseudoinverse, recently we have introduced the concept of the Lagrangian Jacobian inverse and the Lagrangian Jacobian motion planning algorithm that results form solving a constrained minimization of the Lagrange-type objective function Tchoń, Góral & Ratajczak (2015). In this paper we examine further this inverse. Specifically, we begin a study of its singularities, and prove formally its "trajectory attraction" feature discovered in the mentioned reference. Performance of the Lagrangian Jacobian motion planning algorithm is demonstrated shortly in the context of a motion planning of the rolling ball. More numerical examples can be found in Tchoń, Góral & Ratajczak (2015).

The organization of this paper is the following. Section 2 introduces basic concepts including the basic control system, the end-point map, the Jacobian, and the Jacobian inverse. The Lagrangian Jacobian inverse is studied in Section 3. Section 4 presents the

† This research was supported by the National Science Centre, Poland, under grant decision No DEC-2013/09/B/ST7/02368.

Lagrangian Jacobian motion planning algorithm. A bound on the distance between initial and final trajectories and controls in the basic system is provided in Section 5. A sample of results of numeric computations is shown in Section 6. The paper is concluded with Section 7. Proofs are collected in Appendix.

2. Basic concepts

We shall study the kinematics of a nonholonomc robotic system represented as a driftless control system with outputs, of the form

$$\begin{cases} \dot{q} = G(q)u = \sum_{i=1}^{m} g_i(q)u_i, \\ y = k(q) = (k_1(q), \dots, k_r(q))^T. \end{cases} \tag{2.1}$$

The symbols $q \in R^n$, $u \in R^m$, and $y \in R^r$ denote, respectively, the state variable, the control vector, and the output vector. The control vector fields $g_i(q)$ and the output component functions $k_j(q)$ in (2.1) are assumed of the class C^∞. The system inputs $u(\cdot) \in \mathcal{X} \subset L^2_m[0,T]$ are Lebesgue square integrable functions defined on a time interval $[0,T]$, such that for any initial state q_0 and any control function $u(\cdot) \in \mathcal{X}$, the state trajectory $q(t) = \varphi_{q_0,t}(u(\cdot))$ exists for every $t \in [0,T]$.

For a fixed initial state q_0 and the control horizon T, the system (2.1) transforms control functions into the system outputs at the time instant T. This transformation is described by the end-point map of the system,

$$K_{q_0,T} : \mathcal{X} \longrightarrow R^r, \quad \text{such that} \quad K_{q_0,T}(u(\cdot)) = k(\varphi_{q_0,T}(u(\cdot))). \tag{2.2}$$

It has been proved by Sontag (1990) that the end-point map (2.2) is differentiable with respect to $u(\cdot)$. Its derivative

$$J_{q_0,T}(u(\cdot))v(\cdot) = \mathcal{D}K_{q_0,T}(u(\cdot))v(\cdot)$$

will be called the Jacobian of the system (2.1). Given a control function $u(\cdot)$ and the corresponding trajectory $q(t) = \varphi_{q_0,t}(u(\cdot))$, the Jacobian is computed on the basis of the linear approximation of the system along $(u(t), q(t))$. Specifically, it has been shown that

$$J_{q_0,T}(u(\cdot))v(\cdot) = C(T)\xi(T) = C(T)\int_0^T \Phi(T,t)B(t)v(t)dt. \tag{2.3}$$

The trajectory

$$\xi(t) = \mathcal{D}\varphi_{q_0,t}(u(\cdot))v(\cdot), \quad \xi(0) = 0, \tag{2.4}$$

comes from a linear control system (the linear approximation to (2.1))

$$\begin{cases} \dot{\xi} = A(t)\xi + B(t)v, \\ \eta(t) = C(t)\xi, \end{cases} \tag{2.5}$$

where $\xi \in R^n$, $v \in R^m$, $\eta \in R^r$ denote the state, control and output variables, and

$$A(t) = \frac{\partial G(q(t))u(t)}{\partial q}, \; B(t) = G(q(t)), \; C(t) = \frac{\partial k(q(t))}{\partial q}.$$

The matrix $\Phi(t,s)$ solves the evolution equation, $\frac{\partial \Phi(t,s)}{\partial t} = A(t)\Phi(t,s)$, for $\Phi(s,s) = I_n$.

In terms of the end-point map, the motion planning problem in the system (2.1) takes the following natural formulation. For a given initial state $q_0 \in R^n$ and a desired point $y_d \in R^r$, find a control function $u_d(\cdot) \in \mathcal{X}$ such that $K_{q_0,T}(u(\cdot)) = y_d$. The motion planning problem is equivalent to the inversion of the end-point map. Using the

Jacobian approach this problem can be reduced to the inversion of the Jacobian that in turn consists in solving numerically a Jacobian differential equation. For completeness, we shall concisely present its derivation. We begin with an arbitrary control function $u_0(\cdot) \in \mathcal{X}$. If this function solves the problem, we are done. Otherwise, a smooth curve $u_\theta(\cdot)$ in $\mathcal{X}$, $\theta \in R$, needs to be chosen, and the error $e(\theta) = K_{q_0,T}(u_\theta(\cdot)) - y_d$ computed. The curve of control functions should make the error decrease exponentially to zero, what means that for a positive number γ there holds

$$\frac{de(\theta)}{d\theta} = -\gamma e(\theta). \tag{2.6}$$

A crucial step is to differentiate the error, that results in a Jacobian differential equation

$$J_{q_0,T}(u_\theta(\cdot))\frac{du_\theta(\cdot)}{d\theta} = -\gamma(K_{q_0,T}(u_\theta(\cdot)) - y_d). \tag{2.7}$$

After employing a right inverse $J^{\#}_{q_0,T}(u(\cdot))$ of the Jacobian, (2.7) is transformed into the functional differential equation

$$\frac{du_\theta(\cdot)}{d\theta} = -\gamma J^{\#}_{q_0,T}(u_\theta(\cdot))(K_{q_0,T}(u_\theta(\cdot)) - y_d) \tag{2.8}$$

underlying the Jacobian motion planning algorithm. A solution of the motion planning problem can be computed as the limit $u_d(t) = \lim_{\theta\to+\infty} u_\theta(t)$ of the trajectory of (2.8). It follows from (2.7) that in order to define a right inverse of the Jacobian, for a given $\eta \in R^r$, one needs to solve with respect to $v(\cdot)$ the Jacobian equation

$$J_{q_0,T}(u(\cdot))v(\cdot) = \eta. \tag{2.9}$$

This equation is usually transformed to an optimization problem with equality constraints. The well known Jacobian pseudoinverse (the Moore-Penrose inverse)

$$v(t) = \left(J^{P\#}_{q_0,T}(u(\cdot))\eta\right)(t) = B^T(t)\Phi^T(T,t)C^T(T)\mathcal{M}^{-1}_{q_0,T}(u(\cdot))\eta.$$

is obtained by minimizing the control energy

$$\min_{v(\cdot)} \int_0^T v^T(t)v(t)dt,$$

subject to the constraint (2.9). The Jacobian pseudoinverse exists on condition that the mobility matrix $\mathcal{M}_{q_0,T}(u(\cdot)) = C(T)\int_0^T \Phi(T,t)B(t)B^T(t)\Phi^T(T,t)dt\, C^T(T)$ of the non-holonomic robotic system has full rank r.

3. Main results

As a generalization of the Jacobian pseudoinverse, a right inverse of the Jacobian may be derived from the minimization of the Lagrange-type objective function

$$\min_{v(\cdot)} \int_0^T \left(\xi^T(t)Q(t)\xi(t) + v^T(t)R(t)v(t)\right) dt, \tag{3.1}$$

where $Q(t) = Q^T(t) \geq 0$ and $R(t) = R^T(t) > 0$, with the equality constraint

$$J_{q_0,T}(u(\cdot))v(\cdot) = C(T)\int_0^T \Phi(T,t)B(t)v(t)dt = \eta.$$

The Lagrange objective function defines a Jacobian inverse that minimizes not only the control, but also the trajectory variations in the linear system (2.5). It follows that the inclusion of the trajectory term allows to solve the primary motion planning problem along with some secondary trajectory shaping problems, like the obstacle avoidance problem. This Jacobian inverse will be referred to as the Lagrangian Jacobian inverse. The following theorem has been stated in Tchoń, Góral & Ratajczak (2015).

THEOREM 1. *For a fixed control function $u(\cdot)$, the Lagrangian Jacobian inverse is a map $J^{L\#}_{q_0,T}(u(\cdot)) : R^r \longrightarrow \mathcal{X}$ defined as*

$$\left(J^{L\#}_{q_0,T}(u(\cdot))\eta\right)(t) = v(t) = -R^{-1}(t)B^T(t)L(T,t,\eta), \tag{3.2}$$

where

$$L(T,t,\eta) = \, -\psi_{22}(t)\left(\psi_{22}(T) + C^T(T)\mathcal{R}^{-1}_{q_0,T}(u(\cdot))C(T)\psi_{32}(T)\right)^{-1} C^T(T)\mathcal{R}^{-1}_{q_0,T}(u(\cdot))\eta. \tag{3.3}$$

The matrix function $\Psi(t) = [\psi_{ij}(t)]$, $i,j = 1,2,3$, satisfies a system of linear differential equations

$$\dot{\Psi}(t) = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) & 0 \\ -Q(t) & -A^T(t) & 0 \\ D(t)Q(t) & 0 & A(t) \end{bmatrix} \Psi(t), \tag{3.4}$$

with initial condition $\psi_{ij}(0) = \delta_{ij}I_n$, δ_{ij} denoting the Kronecker delta. The matrix $D(t)$ solves the Lyapunov equation

$$\dot{D}(t) = B(t)R^{-1}(t)B^T(t) + A(t)D(t) + D(t)A^T(t), \tag{3.5}$$

initialized at $D(0) = 0$. The mobility matrix is equal to

$$\mathcal{R}_{q_0,T}(u(\cdot)) = C(T)D(T)C^T(T) = C(T)\int_0^T \Phi(T,t)B(t)R(t)B^T(t)\Phi^T(T,t)dtC^T(T). \tag{3.6}$$

In case of the identity output function the Lagrangian Jacobian inverse simplifies.

PROPOSITION 1. *If the output function of the system (2.1) equals the identity then the function (3.3) takes the form*

$$L(T,t,\eta) = \psi_{22}(t)\psi_{12}^{-1}(T)\eta. \tag{3.7}$$

The next Proposition specifies further this conclusion.

PROPOSITION 2. *Suppose that the output function is the identity function. Define an associated mobility matrix as*

$$\mathcal{S}_{q_0,T}(u(\cdot)) = \int_0^T \Phi(T,t)\mathcal{R}_{q_0,t}(u(\cdot))Q(t)\mathcal{R}^T_{q_0,t}(u(\cdot))\Phi^T(T,t)dt. \tag{3.8}$$

If the mobility matrix and the associate mobility matrix are invertible then

$$L(T,t,\eta) = \psi_{22}(t)\psi_{22}^{-1}(T)\mathcal{R}^{-1}_{q_0,T}(u(\cdot))(\mathcal{S}_{q_0,T}(u(\cdot))\mathcal{R}^{-1}_{q_0,T}(u(\cdot)) + I_n)\eta, \tag{3.9}$$

Proofs of both these propositions are included in Appendix.

4. Motion planning algorithm

The Lagrangian Jacobian inverse (3.3) constitutes the Lagrangian Jacobian motion planning algorithm, Tchoń, Góral & Ratajczak (2015). This algorithm relies on solving for the curve $u_\theta(t)$ the following set of differential-algebraic equations, starting from the initial control $u_0(t)$, and with initial conditions $q_\theta(0) = q_0$, $D_\theta(0) = 0$, $\psi_{\theta ij}(0) = \delta_{ij} I_n$.

$$\begin{cases} \dot{q}_\theta(t) = G(q_\theta(t))u_\theta(t), \\ \dot{D}_\theta(t) = B_\theta(t)R_\theta^{-1}(t)B_\theta^T(t) + A_\theta(t)D_\theta(t) + D_\theta(t)A_\theta^T(t), \\ \dot{\Psi}_\theta(t) = \begin{bmatrix} A_\theta(t) & -B_\theta(t)R_\theta^{-1}(t)B_\theta^T(t) & 0 \\ -Q_\theta(t) & -A_\theta^T(t) & 0 \\ D_\theta(t)Q_\theta(t) & 0 & A_\theta(t) \end{bmatrix} \Psi_\theta(t), \\ \frac{du_\theta(t)}{d\theta} = -\gamma R^{-1}(t)B_\theta^T(t)\psi_{\theta 22}(t)\left(\psi_{\theta 22}(T) + \right. \\ \left. C_\theta^T(T)\mathcal{M}_{q_0,T}^{-1}(\theta)C_\theta(T)\psi_{\theta 32}(T)\right)^{-1} C_\theta^T(T)\mathcal{M}_{q_0,T}^{-1}(\theta)e(\theta), \\ A_\theta(t) = \frac{\partial(G(q_\theta(t))u_\theta(t))}{\partial q},\ B_\theta(t) = G(q_\theta(t)),\ C_\theta(t) = \frac{\partial k(q_\theta(t))}{\partial q}, \\ \mathcal{M}_{q_0,T}(\theta) = C_\theta(T)D_\theta(T)C_\theta^T(T),\ e(\theta) = y_\theta(T) - y_d = k(q_\theta(T)) - y_d, \end{cases} \tag{4.1}$$

Given the trajectory $u_\theta(\cdot)$, the solution of the motion planning problem is obtained in the limit $u(t) = \lim_{\theta\to+\infty} u_\theta(t)$.

5. Trajectory and control bound

It is easily seen that a substitution of $v_\theta(t) = \left(J_{q_0,T}^{L\#}(u(\cdot))(K_{q_0,T}(u_\theta(\cdot)) - y_d)\right)(t)$ into the equation (2.8) results in the identity

$$\frac{du_\theta(t)}{d\theta} = -\gamma v_\theta(t). \tag{5.1}$$

Now, let $q_\theta(t) = \varphi_{q_0,t}(u_\theta(\cdot))$ denote a trajectory of the system (2.1) steered by the control function $u_\theta(t)$. Then, a differentiation with respect to θ yields

$$\frac{dq_\theta(t)}{d\theta} = \mathcal{D}\varphi_{q_0,t}(u_\theta(\cdot))\frac{du_\theta(\cdot)}{d\theta}, \tag{5.2}$$

what, after invoking (2.4) and (5.1), gives

$$\frac{dq_\theta(t)}{d\theta} = -\gamma \xi_\theta(t). \tag{5.3}$$

The following result holds for positive and independent of θ matrices $Q(t)$ and $R(t)$.

Proposition 3. *The distance between the current and the initial trajectories and controls in the system steered by the Lagrangian Jacobian motion planning algorithm is upper bounded as follows*

$$\int_0^T (q_\theta(t) - q_0(t))^T Q(t)(q_\theta(t) - q_0(t))dt + \int_0^T (u_\theta(t) - u_0(t))^T R(t)(u_\theta(t) - u_0(t))dt \leq \\ \gamma^2\theta \int_0^\theta \int_0^T \left(\xi_\alpha^T(t)Q(t)\xi_\alpha(t) + v_\alpha^T(t)R(t)v_\alpha(t)\right) dt d\alpha. \tag{5.4}$$

Proof of this proposition is deferred to Appendix.

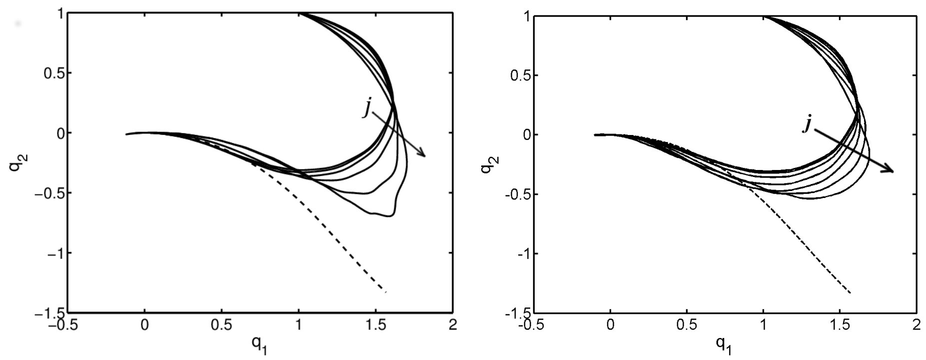

FIGURE 1. Trajectories of the rolling ball in XY plane: $Q_1(t)$ (left) and $Q_2(t)$ (right)

6. Computations

For the kinematics of a rolling ball a motion planning problem has been solved for a constant matrix $R(t) = I_2$ and two series of matrices $Q_1(t) = 10^j I_5$ and $Q_2(t) = 10^j A(t)A^T(t)$, $j = -1, -0.5, 0, 0.5, 1, 1.5$, where $A(t)$ is defined by (2.5). Growing weights increase the role of the trajectory term in the objective function (3.1). Figure 1 reveals that the ball trajectories are attracted by the initial trajectory.

7. Conclusion

This paper presents a new motion planning algorithm for nonholonomic robotic systems, based on the Lagrangian Jacobian inverse. Computer simulations indicate potential applications of this algorithm to shaping the systems trajectories.

8. Appendix

8.1. *Proof of Proposition 1*

We have $C(T) = I_n$, so the equality (3.3) simplifies to

$$L(T, t, \eta) = -\psi_{22}(t) \left(\psi_{22}(T) + \mathcal{R}^{-1}_{q_0,T}(u(\cdot))\psi_{32}(T)\right)^{-1} \mathcal{R}^{-1}_{q_0,T}(u(\cdot))\eta, \tag{8.1}$$

where $\mathcal{R}_{q_0,t}(u(\cdot)) = D(t) = \int_0^t \Phi(t,s)B(s)R(s)B^T(s)\Phi^T(t,s)dt$. Equivalently, (8.1) reads as

$$L(T, t, \eta) = -\psi_{22}(t) \left(\mathcal{R}_{q_0,T}(u(\cdot))\psi_{22}(T) + \psi_{32}(T)\right)^{-1} \eta. \tag{8.2}$$

For any $t \in [0, T]$, let

$$X(t) = \mathcal{R}_{q_0,t}(u(\cdot))\psi_{22}(t) + \psi_{32}(t), \tag{8.3}$$

so

$$L(T, t, \eta) = -\psi_{22}(t)X(T)\eta. \tag{8.4}$$

Differentiation of (8.3) with respect to t yields

$$\dot{X}(t) = \dot{\mathcal{R}}_{q_0,t}(u(\cdot))\psi_{22}(t) + \mathcal{R}_{q_0,t}(u(\cdot))\dot{\psi}_{22}(t) + \dot{\psi}_{32}(t). \tag{8.5}$$

From (3.5) and (3.4), it follows that

$$\dot{\mathcal{R}}_{q_0,t}(u(\cdot)) = B(t)R^{-1}(t)B^T(t) + A(t)\mathcal{R}_{q_0,T}(u(\cdot)) + \mathcal{R}_{q_0,T}(u(\cdot))A^T(t) \quad \text{as well as}$$

$$\dot{\psi}_{22}(t) = -Q(t)\psi_{12}(t) - A^T(t)\psi_{22}(t) \quad \text{and} \quad \dot{\psi}_{32}(t) = \mathcal{R}_{q_0,t}(u(\cdot))Q(t)\psi_{12}(t) + A(t)\psi_{32}(t).$$

A substitution into (8.5) results in the matrix differential equation

$$\dot{X}(t) = B(t)R^{-1}(t)B^T(t)\psi_{22}(t) + A(t)X(t), \quad X(t) = 0. \tag{8.6}$$

On the other hand, from (3.3) we deduce

$$\dot{\psi}_{12}(t) = A(t)\psi_{12}(t) - B(t)R^{-1}(t)B^T(t)\psi_{22}(t), \quad \psi_{12}(0) = 0. \tag{8.7}$$

Summation of (8.6) and (8.7) leads to the identity

$$\frac{d}{dt}(X(t) + \psi_{12}(t)) = A(t)(X(t) + \psi_{12}(t)), \quad (X + \psi_{12})(0) = 0. \tag{8.8}$$

From the uniqueness of solutions of this differential equation one gets $X(t) + \psi_{12}(t) = 0$, so

$$X(t) = -\psi_{12}(t). \tag{8.9}$$

Finally, a substitution of $X(T)$ into (8.4) produces the identity (3.7).

8.2. *Proof of Proposition 2*

We begin with an observation that from the equation (3.4), it follows that

$$L(t) = \psi_{21}(t)\xi(0) + \psi_{22}(t)L(0) + \psi_{23}(t)P(0),$$

so, using the initial conditions $\xi(0) = 0$ and $P(0) = 0$, one gets

$$L(t) = \psi_{22}(t)L(0). \tag{8.10}$$

Since $\psi_{22}(t)$ is a transition matrix for the variable $L(t)$, it must be invertible. Next, assuming the zero initial condition $\psi_{32}(0) = 0$, the solution of the differential equation

$$\dot{\psi}_{32}(t) = \mathcal{R}_{q_0,t}(u(\cdot))Q(t)\psi_{12}(t) + A(t)\psi_{32}(t) \quad \text{takes the form}$$

$$\psi_{32}(t) = \int_0^t \Phi(t,s)\mathcal{R}_{q_0,s}(u(\cdot))Q(s)\psi_{12}(s)ds.$$

From the identities (8.9) and (8.3) one deduces that

$$\psi_{12}(t) = -\mathcal{R}_{q_0,t}(u(\cdot))\psi_{22}(t) - \int_0^t \Phi(t,s)\mathcal{R}_{q_0,s}(u(\cdot))Q(s)\psi_{12}(s)ds, \quad \text{i.e.}$$

$$\psi_{12}(T) = -\mathcal{R}_{q_0,T}(u(\cdot))\psi_{22}(T) - \int_0^T \Phi(T,s)\mathcal{R}_{q_0,s}(u(\cdot))Q(s)\psi_{12}(s)ds. \tag{8.11}$$

It is easily seen that the integral equation (8.11) can be solved as

$$\psi_{12}(s) = \mathcal{R}^T_{q_0,s}(u(\cdot))\Phi^T(T,s)\mathcal{S}^{-1}_{q_0,T}(u(\cdot))(-\psi_{12}(T) - \mathcal{R}_{q_0,T}(u(\cdot))\psi_{22}(T)), \tag{8.12}$$

where

$$\mathcal{S}_{q_0,T}(u(\cdot)) = \int_0^T \Phi(T,s)\mathcal{R}_{q_0,s}(u(\cdot))Q(s)\mathcal{R}^T_{q_0,s}(u(\cdot))\Phi^T(T,s)ds$$

denotes the associated mobility matrix (3.8). Having computed (8.12) at $s = T$, we obtain

$$\left(I_n + \mathcal{R}^T_{q_0,T}(u(\cdot))\mathcal{S}^{-1}_{q_0,T}(u(\cdot))\right)\psi_{12}(T) = -\mathcal{R}^T_{q_0,T}(u(\cdot))\mathcal{S}^{-1}_{q_0,T}(u(\cdot))\mathcal{R}_{q_0,T}(u(\cdot))\psi_{22}(T).$$

We have already shown that $\psi_{22}(T)$ is invertible, so on assumption that the mobility matrix and the associate mobility matrix are so, we deduce invertibility of both the matrices on the left hand side of the above identity. Solving this for $\psi_{12}(T)$ yields

$$\psi_{12}(T) = -\left(I_n + \mathcal{R}^T_{q_0,T}(u(\cdot))\mathcal{S}^{-1}_{q_0,T}(u(\cdot))\right)^{-1}\mathcal{R}^T_{q_0,T}(u(\cdot))\mathcal{S}^{-1}_{q_0,T}(u(\cdot))\mathcal{R}_{q_0,T}(u(\cdot))\psi_{22}(T).$$

In conclusion, by (3.7) and using symmetry of the mobility matrix, we arrive at (3.9). Notice that the inveritibility of $\psi_{12}(T)$ justifies correctness of the formula (3.7).

8.3. *Proof of Proposition 3*

From (5.3) and (5.1) we infer that

$$q_\theta(t) - q_0(t) = -\gamma\int_0^\theta \xi_\alpha(t)d\alpha \quad \text{and} \quad u_\theta(t) - u_0(t) = -\gamma\int_0^\theta v_\alpha(t)d\alpha.$$

In consequence of the above identities, we compute

$$\left(q_\theta(t) - q_0(t)\right)^T Q(t)\left(q_\theta(t) - q_0(t)\right) = \gamma^2 \left(\int_0^\theta \xi_\alpha(t)d\alpha\right)^T Q(t)\int_0^\theta \xi_\beta(t)d\beta \tag{8.13}$$

and

$$\left(u_\theta(t) - u_0(t)\right)^T R(t)\left(u_\theta(t) - u_0(t)\right) = \gamma^2 \left(\int_0^\theta v_\alpha(t)d\alpha\right)^T R(t)\int_0^\theta v_\beta(t)d\beta. \tag{8.14}$$

In what follows we shall focus on the formula (8.13), the other formula can be transformed analogously. Let $Q^{1/2}(t)$ denote a square root of the matrix $Q(t)$, such that $Q(t) = Q^{1/2}(t)Q^{1/2\,T}(t)$. To simplify notations, set $p_\epsilon(t) = \gamma Q^{1/2\,T}(t)\xi_\epsilon(t)$. Then, the right hand side of (8.13) can be written down as $RHS = \left(\int_0^\theta p_\alpha(t)d\alpha\right)^T \int_0^\theta p_\beta(t)d\beta$. Now, we have

$$\begin{aligned} RHS = \int_0^\theta p_\alpha^T(t)d\alpha\int_0^\theta p_\beta(t)d\beta \le \left|\int_0^\theta p_\alpha^T(t)d\alpha\int_0^\theta p_\beta(t)d\beta\right| \le \\ \left|\left|\int_0^\theta p_\alpha^T(t)d\alpha\right|\right| \left|\left|\int_0^\theta p_\beta(t)d\beta\right|\right| \le \int_0^\theta ||p_\alpha(t)||d\alpha \int_0^\theta ||p_\beta(t)||d\beta = \\ \int_0^\theta\int_0^\theta ||p_\alpha(t)||||p_\beta(t)||d\alpha d\beta \le \frac{1}{2}\int_0^\theta\int_0^\theta \left(||p_\alpha(t)||^2 + ||p_\beta(t)||^2\right)d\alpha d\beta \le \\ \frac{1}{2}\theta\left(\int_0^\theta ||p_\alpha(t)||^2 d\alpha + \int_0^\theta ||p_\beta(t)||^2 d\beta\right) = \theta\int_0^\theta ||p_\alpha(t)||^2 d\alpha = \theta\int_0^\theta p_\alpha^T(t)p_\alpha(t)d\alpha. \end{aligned}$$

Finally, after substitution for $p_\alpha(t)$, and using (8.13) we conclude that

$$\left(q_\theta(t) - q_0(t)\right)^T Q(t)\left(q_\theta(t) - q_0(t)\right) \le \gamma^2\theta\int_0^\theta \xi_\alpha^T(t)Q(t)\xi_\alpha(t)d\alpha. \tag{8.15}$$

Along the same lines we deduce

$$\left(u_\theta(t) - u_0(t)\right)^T R(t)\left(u_\theta(t) - u_0(t)\right) \le \gamma^2\theta\int_0^\theta v_\alpha^T(t)R(t)v_\alpha(t)d\alpha. \tag{8.16}$$

By summing up sidewise the inequalities (8.15) and (8.16), and integrating from $t = 0$ to $t = T$, we obtain the bound (5.4).

REFERENCES

Chitour, Y. 2006 A continuation method for motion-planning problems. *ESAIM Contr. Optim. Calc. Var.* **12**, 139–168.

Chitour, Y. & Sussmann, H.-J. 1998 Motion planning using the continuation method. In *Essays on Mathematical Robotics* (ed. Baillieul, J., S. S. Sastry & H.-J. Sussmann), pp. 91–125. Springer.

Divelbiss, A., Seereeram, S. & Wen, J.-T. 1998 Kinematic path planning for robots with holonomic and nonholonomic constraints. Op. cit., pp. 127–150.

Sontag, E. D. 1985 *Mathematical Control Theory*. Springer.

Sussmann, H.-J. 1992 New differential geometric methods in nonholonomic path finding. In *Systems, Models, and Feedback* (ed. Isidori, A. & T. J. Tarn), pp. 365–384. Birkhäuser.

Tchoń, K. & Jakubiak, J. 2003 Endogenous configuration space approach to mobile manipulators: a derivation and performance assessment of Jacobian inverse kinematics algorithms. *Int. Journal of Control* **76**, 1387–1419.

Tchoń, K., Góral, I. & Ratajczak, A. 2015 Jacobian motion planning of nonholonomic robots. The Lagrangian Jacobian algorithm. *Proc. 10th Int. Workshop on Robot Motion and Control*, Poznań, Poland, July 6–8.