Music of the Primes; a talk by Marcus du Sautoy presented to the IMA 40th Anniversary Conference

Not being a number theorist or pure mathematician, and certainly not a musician, it was with some trepidation and hesitancy that I agreed to be a 'scribe' for the talk by Marcus du Sautoy (MdS) on 'The Music of the Primes'. My anxieties proved unfounded, however, for the talk was entertaining and informative whilst also being accessible to a novice such as myself.

MdS began by showing that examples of prime numbers–in other words numbers (excluding 1) which are only divisible by themselves and unity–occur all the time in the world around us. Real Madrid clearly have an interest in prime number theory: David Beckham wears number 23, Zidane 5 and Ronaldo 11. It also appears that nature has an interest, for the cicada lives underground for 17 years before emerging for a mere six weeks to lay eggs, mate and die. The cicada is not the only species to have this behaviour; it is believed that prime number periods of dormancy may help such creatures to avoid predators. Hollywood also has an interest in prime numbers; in the horror film 'The Cube', prime numbered rooms contained grisly booby traps. Indeed, MdS showed a particularly graphic scene from the film to persuade us of the importance of knowing our primes! For the mathematician, however, the significance of the primes lies in the famous Fundamental Theorem of Arithmetic which states that every integer can be decomposed uniquely into a product of primes. Prime numbers can, therefore, be seen as the building blocks, or 'atoms', of the integers.

Yet, for the most part, the primes remain a mystery and enigma. Some of the oldest–and still unsolved–questions in mathematics concern prime numbers: the twin prime conjecture, for example, which asserts that there are infinitely many prime numbers such that p and $p+2$ are both prime, or the Goldbach conjecture which asserts that every even number ≥ 4 can be expressed as the sum of at most two primes. Gauss is accredited as saying that 'Any child can pose questions concerning the primes that the wisest mathematician cannot answer'.

MdS focussed his attention on the age-old question of the distribution of the prime numbers. I guess that many mathematicians have looked at the sequence of primes as a student and wondered whether there was any rhyme or reason to them; given the first n primes is there a formula for the next prime in the sequence? It seems that this is true for most sequences we meet at school, for example the triangle numbers

$$1, 3, 6, 10, 15, \ldots, \frac{n(n+1)}{2}, \tag{1}$$

and the Fibonacci numbers

$$1, 1, 2, 3, 5, \ldots \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\} \tag{2}$$

Is something similar true of the prime numbers as well? Or is their distribution so unpredictable that they are as random as the lottery numbers?

Prime numbers are not just an abstract pure mathematical idea of little significance, but are of fundamental importance in many areas of the 'real world'. In Public Key Cryptography, for example, there are private keys which are primes p and q; a number $N = pq$, which is made publicly known, is used to encrypt (or 'scramble') the document. However, it is only through knowledge of the prime factors p and q that one can decipher the document. Thus, given a number N, if one could determine the (unique) factors p and q (without the laborious task of testing every combination of primes $\leq \sqrt{N}$) then one could break the code and steal everybody's bank details! This would clearly be a calamity for business and commerce. Fortunately, N is usually hundreds of digits long and so to test all the possibilities–even on the fastest computer available–would take so long that the sought-after information would become obsolete; so we can sleep soundly at the present time! However, if a general formula for the primes is found, then this may well lead to an algorithm for finding p and q and so we would have reason to remain awake!

MdS moved on to give the audience a brief review of the history of prime number theory. Since Euclid–in about 300BC–showed that there were infinitely many primes, many mathematicians have investigated this issue. However, it was not until the work of Gauss in 1849 that much more progress was made. Gauss' investigations appear to have begun in 1791 when he was 14. At that time he considered the behaviour of $\pi(n)$–the number of primes less than or equal to n — as $n \to \infty$. By examining graphs of $\pi(n)$ against n he conjectured the asymptotic relation

$$\pi(n) \to \frac{N}{\log N} \quad \text{as } N \to \infty \tag{3}$$

However, Gauss did not prove this result–often referred to as the prime number theorem–and for more than a century, it remained possible (but unlikely) that there was a point beyond which $\pi(n)$ might diverge in some other way.

It was not until Bernard Riemann and his memoir of 1859 'Über die Anzahl der Primzahlen unter einer gegenbenen Grösse' [2] that any deeper insights could be gained into the distribution of the primes. Riemann's work concerned the now famous (Riemann) zeta function, $\zeta(s)$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \tfrac{1}{n^s} \quad \text{for } s > 1. \tag{4}$$

In fact, this function had been discovered and investigated earlier by Euler (approximately 1740). Euler noticed that for $s > 1$, $\zeta(s)$ could be expressed as a convergent infinite product over the prime numbers p, namely

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{5}$$

This slightly surprising result follows by expressing $(1 - p^{-s})^{-1}$ as a Binomial Series, expanding out the infinite product and recalling the Fundamental Theorem of Arithmetic. The Riemann zeta function has many interesting properties of its own, for example,

$$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}|B_{2k}|}{(2k)!} \quad k = 1, 2, 3, \ldots, \tag{6}$$

where B_{2k} are the Bernoulli numbers ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$). The fact that $\zeta(s)$ has a simple pole at $s = 1$ can be used to give a very brief proof of the infinitude of the primes; because

$$\lim_{s \to 1+} \zeta(s) = \infty \tag{7}$$

and the only way that the product on the right hand side of equation (5) can be divergent is if there are infinitely many primes!

Riemann's genius was to consider $\zeta(s)$ as a function of the complex variable s, and this approach led to his insights into the prime numbers. Evidently, $\zeta(s)$ is convergent for $\Re(s) > 1$ but Riemann was able to analytically continue $\zeta(s)$ into the whole complex plane and in so doing he discovered a host of properties. Not least among these was Riemann's explicit formula which connects the distribution of the prime numbers to the zeros of the Riemann zeta function. MdS explained how this formula dictates that the zeros of the Riemann zeta function operate like the modes or harmonics of a musical instrument. So, just as a guitar sound is composed of a series of modes which are allowed to propagate down the string, the distribution of the primes is composed of a series whose modes correspond to the zeros of $\zeta(s)$; hence 'The Music of the Primes'. Mathematically speaking, the idea has close parallels to Fourier analysis where a discontinuous distribution such as the sawtooth graph can be thought of as the limit of a Fourier series. Likewise, the jagged and unpredictable distribution of the primes can be expressed as a series involving the zeros of $\zeta(s)$.

Not only did Riemann's work lead Hadamard and de la Vallée Poisson to prove (independently!) the prime number theorem in 1896, but Riemann made a deeper conjecture concerning the zeros of $\zeta(s)$. The famous Riemann Hypothesis states that the zeros of $\zeta(s)$ all lie on the line $\Re(s) = 1/2$. The conjecture is still unproved today and is regarded as one of the great questions of mathematics. More importantly than this (in these mercenary times), it is one of the Clay Prize Problems; the mathematician who succeeds in proving the Hypothesis will receive a prize of 1 million US dollars! (Quite an increase from the prizes usually awarded for the other great questions of mathematics, which in the experience of the author, vary from a Mars Bar, pint of Beer to a bottle of Champagne as one progresses up the educational ladder.) However, MdS pointed out that the Riemann hypothesis is of far greater significance, for its truth would imply that there is no eficient formula or fast algorithm for generating the primes and thus there would be no fast algorithm for generating the prime factors p, q of a given N. Therefore, the Riemann Hypothesis is of fundamental importance both in pure and applied mathematics.

In this vein, MdS closed his talk by addressing prejudices that exist between pure and applied mathematics; describing himself as a pure mathematician who delights in any application. He discussed the importance of communication and interdisciplinarity between all kinds of mathematicians with their different approaches and experiences. For my part, I have been fascinated by $\zeta(s)$ since I came across it (understanding very, very little) at school and as an applied mathematician I would be delighted if it could be shown to lie at the heart of a practical applied mathematics problem.

For a much better and fuller account of these ideas we refer the reader to MdS's recent book [3]. Two other books which I have found interesting are [1] which gives a straightforward yet comprehensive introduction to the subject of Number Theory and [4] which is a detailed introduction to the Riemann Zeta function. Also a browse of the Internet may yield some interesting revelations concerning prime numbers and their history–it was here that I found a transcription of Riemann original paper [2]!❑

REFERENCES

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