Nonlinear Mathematics in Structural Engineering

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Introduction

Structural engineering is a discipline with a distinguished history in its own right with its landmark monuments and famous personalities from centuries past to the present [1, 2]. Moreover, it is also a discipline that relies on rich nonlinear mathematics as its basis. The aim of this article to show some of the interesting features and practical relevance of nonlinear mathematics in the behaviour of real structures. It is an area of research where the UK has led the way for many years.

In the response of structures under loading there are many different sources of nonlinearities. However, for the purpose herein the various cases can be grouped into two distinct categories: (1) material and (2) geometric nonlinearities. The sources of nonlinear material behaviour can arise from the response where the constitutive law (relating stress to strain) in the elastic range is not linear – termed *nonlinear elasticity*. Materials such as mild structural steel have a linear elastic constitutive law, but other important structural materials such as concrete, aluminium, and alloys of iron such as stainless steel are all examples where the elastic constitutive law is nonlinear. Another route to nonlinearity in the material response can occur even in linear elastic materials when the stress exceeds the so-called *yield stress*; permanent deformation (plasticity) ensues and the constitutive law departs from the initial linear relationship (Fig. 1). For brittle materials, such as cast iron, fracture, rather than plasticity, follows the elastic response; a further example of material nonlinearities governing the mechanical response during failure.

The main focus herein is, however, on geometric nonlinearities that govern structural behaviour when large and possibly sudden deflections are seen, often as a loss of stability when the phenomenon known as *buckling* is triggered. In structural engineering this is most likely in elements in whole or in part compression such as columns and beams. Most rudimentary structural mechanics principles are based on the linear assumptions in that although structures deform, they do so slowly with small deflections. Linearization in this context can be typified by the familiar assumption when dealing with small angles:

$$\sin\theta \approx \theta, \tag{1}$$

which lies as a basis for standard so-called "Engineer's" bending theory that relates how external and internal forces affect a beam's deflection, see Fig. 2(a); in particular, the key relationship in bending theory is that the internal bending moment is directly proportional to the beam's curvature which is assumed to equal the second derivative of the out-of-plane displacement w with respect to x. However, this linear relationship is enhanced when curvatures become moderately large:

$$M = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\left[1+\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^{-3/2} \tag{2}$$

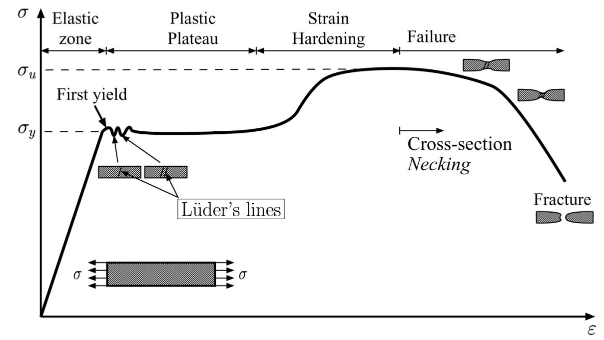

Figure 1: Stress σ vs strain ε sketch for a stretched mild steel bar. Strain is defined as the ratio between extension and the bar's original length. Progressive deformation of the bar is represented along the graph and note the narrow linear elastic range.

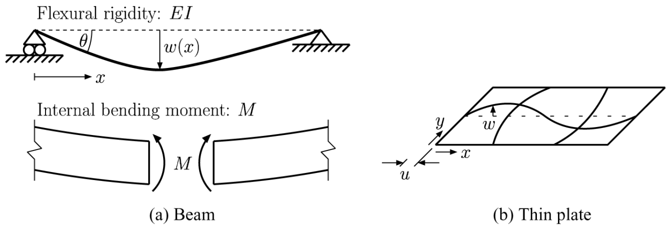

Figure 2: (a) A deflected beam with flexural rigidity EI and internal bending moment M. Note: "flexural rigidity" is essentially the beam's bending stiffness. (b) A thin plate with constrained edges showing in and out of plane displacements u and w respectively.

the term in square brackets becoming significant as the slope of w with respect to x, or more simply the beam's local rotation θ, increases. In thin-walled structures, see Fig. 2(b), the strain in the x-direction ε_x versus displacement relationship is:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{3}$$

which is expressed in terms of a linear term for the in-plane displacement u and a quadratic term accounting for the effect from the out-of-plane displacement w. Geometric nonlinearities such as the term in the square brackets in (2) and the second term in (3) govern whether a structural component can withstand a critical load calculated by linear analysis, whether they can surpass this load significantly, or fail dangerously below it.

Nonlinear buckling

In statics problems it is often more convenient to formulate the governing equations using total potential energy V as opposed to applying Newton's laws of motion to a free-body; V is defined as the sum of the gain in potential energy U and the work done Φ. In structural problems U is *strain energy*, directly analogous to the energy stored while stretching or compressing a spring, and Φ is equal to the load P multiplied by the distance the load moves Δ in the direction of load – this quantity is usually negative as the structure moves in the same direction as the load causing a reduction in V. Therefore, it is more common to write V as follows:

$$V = U - P\Delta. \tag{4}$$

The basis for using V in nonlinear buckling analysis was pioneered principally by Koiter [3]; two essential axioms follow that link V to equilibrium and stability for static systems [4]:

Axiom 1 *A stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the equilibrium of the system.*

Axiom 2 *A complete relative minimum of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the stability of an equilibrium state.*

These axioms say basically that when the first derivative of V vanishes we have equilibrium and the second derivative of V in most cases defines the stability or otherwise of the equilibrium state. However, the interesting cases arise when the second derivative of V vanishes – this defines the *critical* equilibrium, where the structure first buckles ($P = P^{C}$). For example if V for a single degree-of-freedom system is written as a Taylor series with Q being the generalized coordinate and δ being a perturbation, we have:

$$V(Q+\delta) = V(Q) + \frac{dV}{dQ}\delta + \frac{1}{2!}\frac{d^2V}{dQ^2}\delta^2 + \ldots + \frac{1}{n!}\frac{d^nV}{dQ^n}\delta^n + \ldots \tag{5}$$

Axiom 1 states that for equilibrium the first derivative of V vanishes, hence V is rewritten:

$$\begin{aligned} V(Q+\delta) - V(Q) &= \frac{1}{2!}\frac{d^2V}{dQ^2}\delta^2 + \frac{1}{3!}\frac{d^3V}{dQ^3}\delta^3 + \frac{1}{4!}\frac{d^2V}{dQ^4}\delta^4 \\ &+ \ldots + \frac{1}{n!}\frac{d^nV}{dQ^n}\delta^n + \ldots, \end{aligned} \tag{6}$$

and this implies that the right-hand side of (6) has to be positive for V to be minimum and therefore the equilibrium state to be stable by Axiom 2.

Now for systems that assume linear elasticity and small displacements, the highest order term in V could only be quadratic in Q and so the highest derivative of V with respect to Q that could be non-zero is only the second one. Once that term is zero, which would imply a change of stability in the equilibrium state, any perturbation δ would have no measurable effect on the system. Therefore, further information about the stability of the new equilibrium state cannot be obtained. For this "post-buckling" information to be established, nonlinearities, in this case those arising from large deflections, need to be retained in the model as they would allow non-trivial higher derivatives in V to dominate the series when the second derivative vanishes; in systems with more degrees of freedom an analogous situation exists with the Taylor series involving more generalized coordinates.

Bifurcations: Stability and instability

Post-buckling or nonlinear buckling theory gives the engineer the information whether the system has any residual load carrying capacity once the critical load P^{C} is reached. In the most common case there is theoretically no displacement out of the plane of loading until the load P reaches P^{C}, i.e. the geometry of the system has no imperfections leading to secondary stresses from eccentricities. When this occurs the system usually encounters *a pitchfork bifurcation point*, the leading non-zero term in the Taylor series of V has an even power; when this term is positive we have a *stable* or *supercritical buckling* scenario (Fig. 3), and if this term is negative we have an *unstable* or *subcritical buckling* scenario (Fig. 4). A less common case occurs when the leading term in V at P^{C} has an odd power, in this case the point is classified as a *transcritical bifurcation point* and *asymmetric buckling* is triggered which is broadly similar to the unstable case in terms of the implications for the practical structural response. It is worth noting here that at undergraduate level, most discussion of buckling in engineering courses is confined to the so-called Euler strut (or column), a model of which can easily be made by compressing a plastic ruler. Although this component is intrinsically stable, its post-buckling strength is insignificant and it is one of the very few examples where linearization gives meaningful information to the engineer. Therefore, a graduate structural engineer with a lack of appreciation of the difference between linear and nonlinear buckling may be ignorant of their designs being overly conservative or optimistic in terms of the true strength.

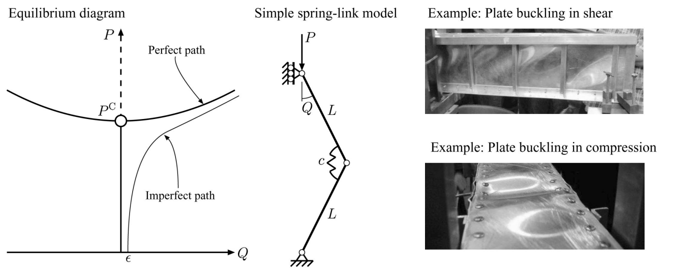

Figure 3: Nonlinear response of a stable buckling system. Left to right: a typical force vs deflection diagram, a simple torque spring–rigid link model that gives a stable response; examples of real components that show this behaviour.

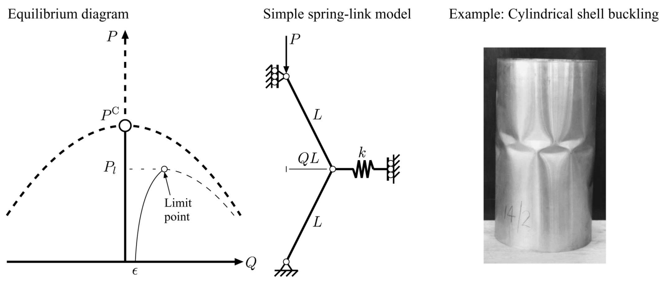

Figure 4: Nonlinear response of an unstable buckling system. Left to right: a typical force vs deflection diagram; a simple longitudinal spring–rigid link model that gives an unstable response; an example of a real component that shows this behaviour.

Structures with imperfections

The *equilibrium diagrams* in Figs. 3 and 4, another term for the force versus displacement diagrams, also show the effect of imperfections on the system's mechanical response. Imperfection sources are varied: manufacturing processes not giving perfectly flat plates, welding giving different properties from place to place in a structural component, elements not aligning correctly, and so on. Examining Fig. 3, the imperfect equilibrium path increases monotonically and is asymptotic to the perfect case even beyond the critical load P^{C}, therefore stable buckling structures can still be loaded beyond P^{C}. Moreover, because the imperfection size is independent of the theoretical maximum load, this type of system is said to be insensitive to imperfections but only if the material remains linearly elastic: if it softens or goes plastic then the situation changes significantly.

Conversely, if Fig. 4 is considered, the imperfect equilibrium path shows the load increasing at first but then hits a maximum – *limit point or saddle–node bifurcation point* – that is below P^{C}, and the rest of the path is still asymptotic to the perfect case. From this we can infer that unstable buckling structures can never attain P^{C}. Moreover, the greater the imperfection size, the greater the reduction in the maximum load. Hence we say that even linearly elastic structures that are unstable are *imperfection sensitive* and an approximate mathematical rule can be derived relating the imperfection size ε to the corresponding limit load P_l:

$$\frac{P_l}{P^{C}} = 1 - \alpha\varepsilon^{2/3} \qquad (7)$$

where α depends on the system. The major point is that understanding the nonlinear behaviour can allow the engineer to design a structure to be more efficient if it has stable post-buckling characteristics as allowing it to buckle is less serious – local (plate) buckling in aeronautical structures is commonly allowed at working loads as long as the structural stiffness does not fall below a threshold level. If, however, the structure is intrinsically unstable the engineer would know that the linear critical load could be a gross overestimate of the ultimate strength of the component and either factors of safety would be employed or nonlinear modelling and analysis would be conducted to establish a more accurate strength.

Localization, Periodicity and Cellular Buckling

Once buckled, structures physically have distinctive qualitative features. Examining the photographs in Figs. 3 and 4 it is noteworthy that the stable plated structures shown have a periodically repeating buckle pattern, whereas the unstable cylinder has a buckle pattern that is localized to a small region of the structure. These features are not coincidences, they are general to buckling responses. In fact, if stable structures begin to show localized deformation it means that plastic failure is imminent; the localized deformation region is known as a hinge and collapse ensues.

A helpful model that illustrates the different responses is the ubiquitous strut resting on an elastic foundation (Fig. 5), which has a governing fourth-order ODE:

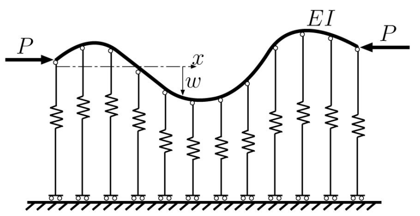

Figure 5: Strut on an elastic foundation with flexural rigidity *EI*, axial load *P*, buckling displacement w. Springs have a nonlinear elastic force–displacement relationship *F*(*w*).

$$EIw'''' + Pw'' + F(w) = 0, \qquad (8)$$

where primes represent differentiation with respect to the axial coordinate x and $F(w)$ relates to the nonlinear foundation force–displacement relationship that can be rewritten with the linear elastic term and $f(w)$ having the nonlinear terms only:

$$F(w) = kw + f(w). \qquad (9)$$

An excellent review of the intricacies of the behaviour of equation (8) can be found in [6]; the discussion herein is confined to the key results and their practical implications. It is also worth noting that an addition of a time derivative of w changes this into a PDE very similar to both the Swift–Hohenberg and the Extended Fisher–Kolmogorov equations [5].

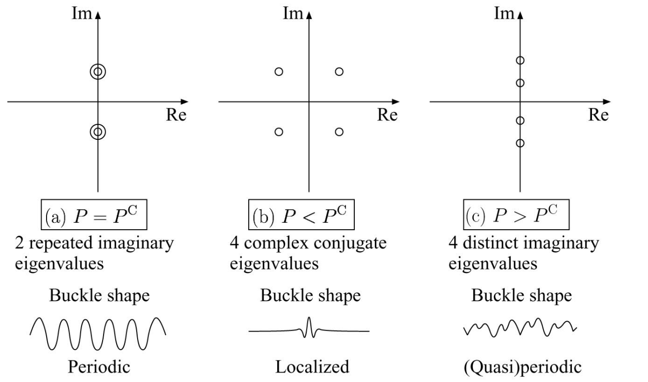

Figure 6: Characteristic eigenvalues of the strut on an elastic foundation. (a) Critical load: periodic, $P^{C} = 2\sqrt{kEI}$; (b) Subcritical: softening *F* gives localization; (c) Supercritical: hardening *F* gives periodicity.

Returning to the static equation (8), the expression for F(w) governs the post-buckling response. A strongly softening foundation, where the force versus displacement slope decreases, for example: $f(w) = cw^2$ or $f(w) = -cw^3$ where $c > 0$, gives a periodic buckling response at P^{C} (Fig. 6(a)) which changes to a modulated pattern in the subcritical range – associated with four complex conjugate eigenvalues (Fig. 6(b)) when solving the linearized differential equation ($f = 0$). As P reduces, a secondary bifurcation changes the response to a localized buckling mode that is the signature of an *homoclinic* connection, i.e. the buckling displacement is basically zero as the boundaries are approached in each direction. Where the localized buckling displacement is significant, the softening nonlinearity in the foundation forces the deflection back to zero. In a long strut, the exact location of the localized buckling is also strongly sensitive to the boundary conditions and in this way it can be said technically to be *spatially chaotic* [7].

A strongly stiffening foundation, where the force versus displacement slope increases, for example: $f(w) = cw^3$ or $f(w) = cw^5$ where $c > 0$, gives a similar response at P^{C}, but the post-buckling now is supercritical and the buckling mode locks into the periodic mode defined initially by the associated four imaginary eigenvalues, (Fig. 6(c)). As the load increases, the system again becomes vulnerable to secondary bifurcations, in this case jumping to a new periodic mode with a different wavelength rather than to a qualitatively different localized mode [8]. The phenomena of *localization* and *mode locking and jumping* are strongly linked to unstable and stable behaviour respectively. In practical structures the softening or the stiffening nonlinearities arise naturally from a variety of geometric sources: continuous supports and large bending curvatures give railway lines and pipelines an unstable response as does the simultaneous triggering of buckling modes – global and local mode interaction being common particularly in axially compressed sandwich structures and cylindrical shells; membrane action from the double curvature in the buckling deformation

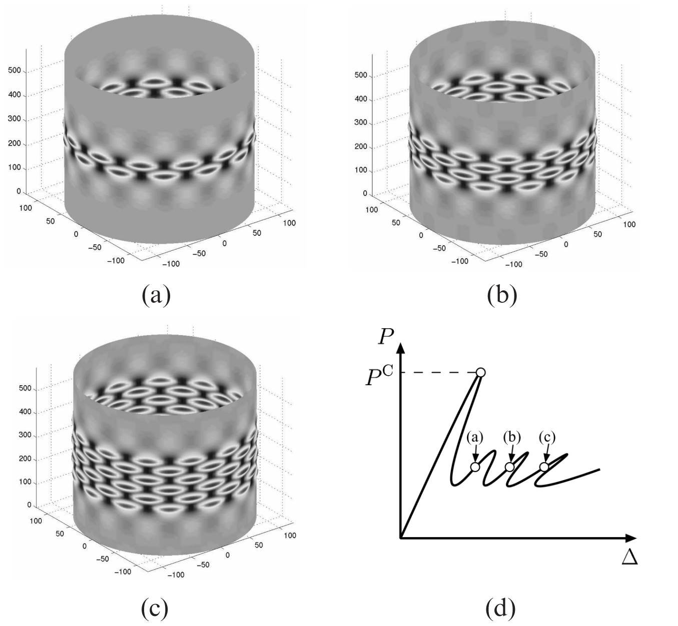

Figure 7: (a)–(c) Sequence of numerical solutions of the cylindrical shell equations: the shading showing the radial displacements [9]. (d) Sketch of the load P vs end-displacement Δ graph showing where each buckle pattern is represented, note that each maximum on this graph represents the appearance of a new buckle "cell".

gives the stable response in plated structures – see Fig. 2(b) – such as those found in flat metal panels in aerospace structures and in bridges with thin-walled cross-sections.

The axially compressed cylindrical shell (Fig. 7) is an example where an initially unstable post-buckling response subsequently restabilizes and then may destabilize again and restabilize again and so on. Here, the initially localized deformation is added to in a modular way where each sequence of destabilization and restabilization adds a cell of localized buckling deformation. Of course, in the limit this cellular deformation would cover the entire structure and the buckling deformation tends to periodicity [9]; this undeformed state to localized buckling to periodic buckling transformation is an example of an *heteroclinic* connection familiar from nonlinear dynamical systems theory [10, 11]. To simulate this response in the strut on foundation model, the foundation function f would need counteracting terms, for example $f(w) = -w^3 + cw^5$ where $c > 0$ but not so large as to dominate the effect of the softening cubic term completely. This type of response can also be seen in the yielding of the steel bar in Fig. 1: the characteristic wiggles near the first yield point signify the appearance of *Lüder's lines* (localized shear deformation lines) the number of which increase as the strain increases and the stress σ oscillates around σ_y before plasticity really takes hold.

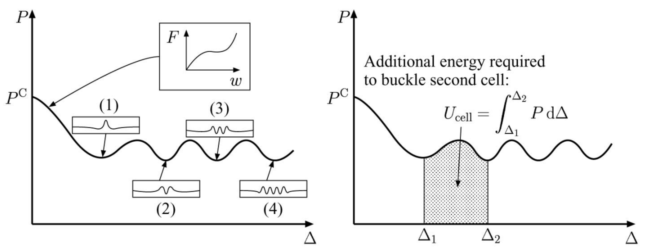

Figure 8: Representation of cellular buckling in the strut on a foundation that softens and then stiffens. Note the number of buckle peaks increasing with each minimum along the load P vs end-displacement Δ plot along with the energy required to buckle new cells.

The cellular buckling response can be taken advantage of practically in, for example, the dissipation of energy from the impact in a car crash. So-called "crumple zones" in cars can essentially be cylinders that are designed to buckle dynamically in a cellular fashion, each buckle cell being associated with a packet of energy being absorbed as represented in Fig. 8. Plenty of other structures show this sequential cellular response including those formed in natural geological processes such as the folding of rock strata from tectonic action [12]; Fig. 9 shows sequences of experiments on compressed layers of paper that simulate the cellular buckling involved in the geological folding of strata into chevrons and concentric or parallel folds respectively. Structural geologists use buckling principles to model such formations as they can give clues to the locations of precious metal and mineral deposits. Nonlinearities here arise from the discontinuous nature of friction between the layers and the large rotations of the layers in the folding geometry.

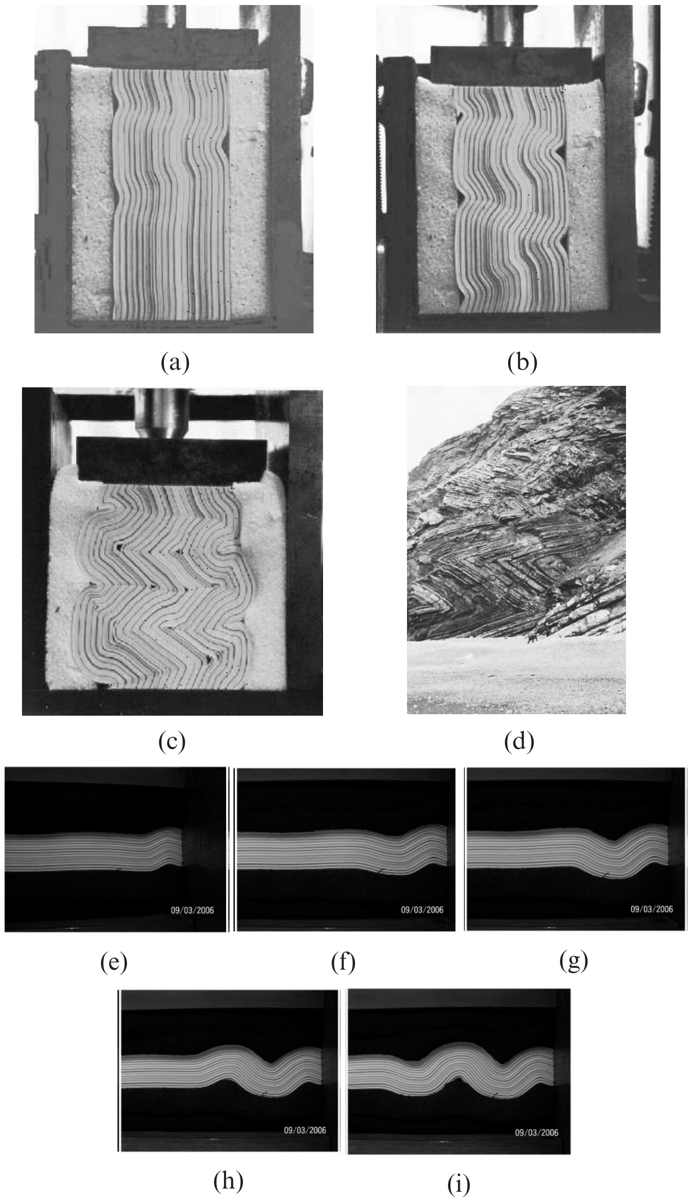

Figure 9: Sequence of experimental photographs showing compressed layers of paper buckling in a cellular fashion to simulate the formation of chevron folds (a–c) and parallel folds (e – i) in geological strata.Photograph (d) shows actual chevron folds in Millook Haven in Cornwall.

Concluding remarks

Although structural engineering is a well established discipline it is also a source of rich nonlinear mathematics at its fundamental level, in which the current article only scratches the surface. Significant developments have arisen from cross-fertilizing with dynamical systems theory from the early 1960s. However, it is not only important for mathematicians to understand and appreciate this, practicing engineers need to be aware of the issues that the naturally occurring nonlinearities in their systems throw at them if they continually wish to improve their understanding of how their designs work and how they can make them more efficient while maintaining safety. Instabilities in equilibrium and sensitivities to imperfections are a couple of vitally important issues that code developers for structural design practice take very seriously; it is perhaps comforting to know that designers follow procedures that have been developed from a robust theoretical basis.❑

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