Westward Ho! Musing on Mathematics and Mechanics

Alan Champneys CMath FIMA, University of Bristol

In the latest of his features from Bristol and the West Country, Alan Champneys considers the growing craze of surfing that features on many west facing beaches in Devon and Cornwall. The journey will take us on through an elementary guide to water wave theory, from pond ripples to bores, illustrating just a few of the mathematical subtleties. We will also examine various myths, such as on the groupings of waves, the naming of rock festivals and one of the most grossly misnamed equations in science.


© Epicstock | Dreamstime.com

Boardmasters

In my last Westward Ho!, I extolled the virtues of days spent at the beach. I grew up in the South-East of England. Trips to the beach involved sand castles, rock scrambling, paddling, eating too much ice cream and losing money on various ingeniously designed slot machines. I can recall jumping waves, but never trying to ride them. Surfing was something that only happened on TV (especially in adverts for a certain brand of men's aftershave) in a world I could only dream of. It was not until I relocated to the West Country that I discovered beachside shops selling wet suits, long boards, body boards, and various fashion accessories.

In August this year my daughter will travel with her friends to Boardmasters in Newquay. This is supposedly a family friendly festival that combines music with surfing and other beach sports. It's a particular take on the modern summer music festival craze that has become ever-so-popular across Europe. The original such festival takes place on a farm in the small village of Pilton near Shepton Mallett. I have never met Michael Eavis, the founder, but he seems like a genuine man of principle. Nevertheless, I would argue that had he not chosen the life of a dairy farmer, social campaigner and music promoter, he could have had a successful career in advertising. 'Shepton Mallet festival' does not have the same ring to it as 'Glastonbury!'

It would also appear that the fame from the festival has done wonders for the small Somerset town that is some 5 or 6 miles from Worthy Farm. I remember visiting about 30 years ago what was a rather sleepy market town, with rather tenuous connections both to Joseph of Arimathea and to the King Arthur legend. Glastonbury has now become a mecca for New Age folk, with shop after shop selling strange smelling salts, crystals and books on magik, crop circles, and the like. I recall sitting in a coffee shop a few years ago next to a jolly family with young children in a somewhat exuberant mood: 'Merlin, Morgana, please play nicely!'

I digress. I never took to surfing. I had two lessons and did not even manage to catch a wave, let alone stand up. It is a mystery to me how my lithe sons and nephews manage to glide with some ease on even quite small waves. I do enjoy body boarding though. I would love to write a piece on the mechanics of surfing, but it seems to me to rely on a complex mixture of rigid body mechanics, boundary layer theory, fluid transport and not a little skill on the part of the surfer. So let us just concentrate on the waves that they try to ride.

A most obvious observation is that the waves breaking on a beach are more spread out and have steeper amplitudes than the periodic waves I see when I drop a stone into a pond. I have often been fascinated by water waves, but I realise there is almost nothing I understand. My PhD supervisor was an internationally acclaimed expert. When I did my postdoc I also became aware of how water wave theory has spawned many problems in non-linear analysis: the steepest wave, rogue waves, surface tension as a singular perturbation, Boussinesq approximations, Stokes waves, long waves, shallow-water approximations, integrability, scattering, inverse scattering . . . and much more. Indeed many great mathematicians including the IMA's current president have studied, and still do, the sequences of crests and shallows that reach our shores. Over the years, I have dabbled in solitary-wave theory, but I would not call myself an expert in fluid mechanics by any stretch of the imagination. So can I overcome my inferiority complex, and try to understand at the least the basics of how surfable waves form? Probably not. I will leave the reader to judge.

The first question I want to understand is what causes the changing amplitudes of the waves that surfers ride on the beach. In particular, is there any truth in the surfer's adage that the seventh wave is always the biggest?

A water wave is more precisely referred to as a form of surface gravity wave. This confused me and made me think that, like the tides, ocean waves are driven by the moon's gravity. They are not; the name 'surface gravity wave' simply refers to the interface between water and the air above it. Ocean waves are initiated by wind, rather like the regular waves you might see on a boating lake in a park on a windy day. But what surfers are really looking for are not waves that are directly formed by the local wind, but a *swell*. That is, a series of waves generated by a distant weather system, which have been carried for a duration of time over a *fetch* of water. Generally speaking, the longer the fetch the more the waves disperse (become well organised into waves of different frequencies travelling at different speeds), and the more they grow in amplitude and become steeper (become less and less like sine waves) and so become more fun to surf.

Actually, the precise mechanism of swell generation in deep-water oceans is not straightforward. There can be many random effects. An orthodox explanation seems to involve at least three steps. First, turbulence in the wind results in random pressure fluctuations to the sea surface causing small-amplitude waves with wavelengths of the order of a few centimetres. A cross wind then amplifies these waves, with non-linear feedback occurring through shear instabilities. Then, random interactions between the waves transfer energy into longer, faster waves. The waves then disperse so that the longest waves travel fastest. So, when a swell first arrives at the shore, the waves tend to be small, long and fast. There then seems to be a sweet spot for surfers some hours later with larger amplitude waves travelling at roughly the same speed, with just a small spread of frequencies.

The further the fetch, the more this effect is exacerbated. It takes of the order of a day for a swell to travel across 1,000 km of sea. This is why surfing beaches on the UK coast typically face west to capture swells that have travelled across the Atlantic. On the west coast of the US, waves typically have even greater amplitude and longer periods between waves, because of the longer fetch across the Pacific.

But why the seventh wave? It is certainly true that swell waves tend to come in modulated groups, this is a natural consequence of linear superposition of waves of different, nearby frequencies. Another example is the principle of 'beating' that we hear when two nearby notes are played simultaneously. So there are intervals of high wave energy, followed by intervals of lower energy. This has the appearance of small packets, or groups of waves (see Figure 1) with waves at the front and back being of smaller amplitude and having longer inter-peak frequency. So the biggest waves are in the middle of the group.

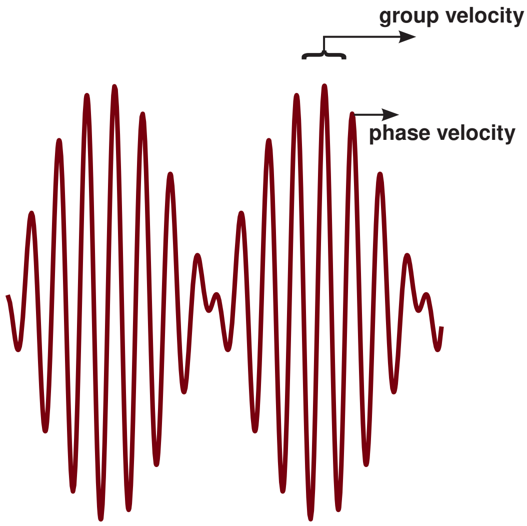

Figure 1: Illustrating the difference between phase and group velocity in a packet or group of waves.

There is more subtlety. However, within each group, waves are constantly forming and disappearing. Taking an individual wave crest it typically forms at the back, has a finite life before disappearing off the front as an inconsequential ripple. This property is due to the difference between phase velocity (the speed of an individual wave crest) and group velocity (the speed of the whole group of waves) of water waves. In theory, at least for small-amplitude waves, phase velocity is twice group velocity. Next time you drop a stone in a pond, look at the different speed of an individual wave crest and the speed at which the circular patch of waves spreads out.

But is the seventh wave always the biggest? This would depend on the number of waves in a group. But that number would depend on the modulation period of the wave packet as well as the dominant underlying frequency of the individual waves. However, as pointed out by one of the US's top meteorologists John Guiney, writing in *Scientific American* [1] there are no set numbers for these two constants. They depend on all kinds of details like the length of the fetch, the severity of the storm causing the waves in the first place, interactions with other winds since the swell was generated, and local bathymetry variation. The fact that waves are constantly moving from the back to the front of a group also means that the notion of what is the seventh wave is not actually well defined. A surfer might look far out to sea, spot the seventh in a group and follow that crest till it comes into the surf zone. But by then that crest may have moved towards the front of the group and become significantly smaller. Nevertheless, there may be some approximate truth in the seventh-wave theory because, at least for long Pacific fetches, the number of waves in a group is typically of the order of 10–20, with 14 or 15 not being uncommon.

Next, why do waves break? Again there are many answers and as far as I know it is still a matter of ongoing research to explain why small spatio-temporal localised patches of white spray can be observed in even moderate seas. Why waves break as they approach the shore though is reasonably well understood. It is all to do with shallow-water theory. As the depth decreases, a balance between non-linear focusing effects and spatial dispersion tends to lead to steeper, more localised waves. The waves get steeper and steeper, until they become too steep to be described by a (weakly) non-linear balance. Continuum theory breaks down. The wave eventually overturns which causes turbulent effects to dominate, leading to the tumbling and crashing that is characteristic of the rolling breakers sought by surfers.

How can we describe mathematically at least the first part of this process, non-linear wave steepening? Again it seems to me that things are rather confusing, because of the many different model equations that can be derived by choosing different distinguished limits of the time and space scales involved (see for example [2] for a discussion of the physical applicability of the various approximations and the dangers of appealing to the wrong mathematical theory). Let me first start from first principles. Much of what follows can be found in the excellent book by Darrigol on the history of fluid mechanics [3].

In fact, we first need to understand how to get a wave equation at all for the free surface deformation of water, from the underlying Navier–Stokes equations of incompressible flow. We shall consider flow in a direction x of fluid at a height y above a flat ocean floor at $y = 0$, and suppose that the free surface when undeformed is at height $y = h$. (To consider the full breaking process, we would need to let h vary slowly with x, but that is beyond the scope of this brief derivation.) Let $w(x, t)$ represent the additional height of the free surface due to a wave. Let $u(x, y, t)$ represent the horizontal velocity and $v(x, y, t)$ the vertical velocity at any point in the fluid. As is common when describing the flow of water, we assume that viscosity can be neglected and that the fluid is irrotational and incompressible. Under such assumptions, it was first shown by Euler that the flow can then be described by a potential function $\phi(x, y, t)$ such that $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$.

The continuity equation says that mass must be conserved in an incompressible fluid. This implies $\partial u/\partial x + \partial v/\partial y = 0$ at every point of the fluid. Taking the potential form of the velocities we obtain the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad \text{for all} \quad 0 < y < h + w. \tag{1}$$

So we are looking for harmonic functions within the water. All the complexity comes when we consider the boundary conditions.

At the bottom of the channel, it is natural to assume that there is no flow escaping:

$$(v =) \quad \frac{\partial \phi}{\partial y} = 0 \quad \text{for} \quad y = 0. \tag{2}$$

The boundary conditions at the top of the channel come from two more physical principles. First there is a *kinematic boundary condition* that a particle on the free surface remains on the free surface. Thus the vertical velocity must be equal to the change of height of the wave

$$(v =) \quad \frac{\partial \phi}{\partial y} = \frac{\partial w}{\partial t} \quad \text{at} \quad y = h + w. \tag{3}$$

The second is a *dynamic boundary condition* that balances forces across the free surface. Ignoring the (weak) effect of surface tension, this leads to the Bernoulli equation,

$$\frac{\partial \phi}{\partial t} + gw + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] = \text{const.} \quad \text{at} \quad y = h + w. \tag{4}$$

Note that the only non-linearity is in the dynamic boundary condition.

From the above as derived by Euler, Lagrange was the first, in 1871, to derive a linear wave equation under certain simplifications. Specifically, suppose that the depth of the channel is small compared to the wavelength λ of a typical wave $h \ll \lambda$ and the height of any wave is small compared to the depth $w \ll h$. Since h is small, it is now supposed that the variation of ϕ in the y direction is weak and so we can write to leading order:

$$\phi = \phi_0(x,t) + y^2\phi_2(x,t). \tag{5}$$

Note that there cannot be a linear term in y, otherwise we could not satisfy the bottom boundary condition (2) at $y = 0$. Feeding this form of ϕ into the Laplace equation (1), we find to leading order in y,

$$\phi_2 = -\frac{1}{2}\frac{\partial^2 \phi_0}{\partial x^2}. \tag{6}$$

The main difficulty is how to simplify the non-linear, dynamic boundary condition (4). If we suppose that the velocities are all of the same order of magnitude as w, and keep only the leading-order terms, this condition becomes

$$\frac{\partial \phi_0}{\partial t} + gw = \text{const.} \quad \text{at} \quad y = h + w, \tag{7}$$

which differentiates to

$$\frac{\partial^2 \phi_0}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad y = h + w. \tag{8}$$

To obtain (8) we have used the kinematic boundary condition (3) to substitute $\mathrm{d}\phi/\mathrm{d}y$ for $\mathrm{d}w/\mathrm{d}t$ on the free surface.

Combining (8) with (5) and keeping only the leading-order terms gives

$$\frac{\partial^2 \phi_0}{\partial t^2} - gh\frac{\partial^2 \phi_0}{\partial x^2} = 0.$$

Differentiation of this equation with respect to t and substitution of $\partial\phi_0/\partial t = gw+$ const. from (7) we find that the height $w(x,t)$ satisfies the familiar linear wave equation:

$$\frac{\partial^2 w}{\partial t^2} - gh\frac{\partial^2 w}{\partial x^2} = 0,$$

with wavespeed $c_0 = \sqrt{gh}$.

But what happens when the wave amplitude gets larger? Now there are many possible choices of approximations, but I am going to describe just one, which leads to the much cited Korteweg–de Vries (KdV) equation. We begin by again expanding the velocity potential ϕ as in (5), from which we again obtain (6). It is helpful at this point to non-dimensionalise, using a characteristic wavelength λ. Specifically we write

$$\bar{x} = \frac{x}{h}, \quad \bar{y} = \frac{y}{\lambda}, \quad \bar{t} = t\frac{c_0}{\lambda}, \quad \bar{w} = \frac{w}{a}, \quad \bar{\phi} = \frac{\phi}{c_0}, \tag{9}$$

where a is the maximum amplitude of the waves we wish to describe. Dropping the overbars for ease of presentation, it is useful to introduce two small parameters

$$\varepsilon = (a/h) \quad \text{and} \quad \delta = (h/\lambda)^2,$$

and look for waves for which ε and δ are small.

Equations (5) and (6) written in the scaled variables (9) now imply that on the free surface $y = 1 + \varepsilon w$,

$$\phi = F - (1/2)(1+\varepsilon w)^2\delta F_{xx} + (1/6)\delta^2 F_{xxxx} + \mathcal{O}(3) \tag{10}$$

where $\mathcal{O}(3)$ means terms that are at least of cubic order in ε and δ, and $F(x,t)$ is the scaled version of what was previously called ϕ_0. From (10) we obtain

$$u = \phi_x = f - (\delta/2)f_{xx} + \mathcal{O}(2), \qquad \text{at} \quad y = 1 + \varepsilon w, \tag{11}$$

where $f = F_x$. From differentiation again, using (6) evaluated at a height y we similarly obtain

$$v = \phi_y = -\delta(1+\varepsilon w)f_x + \frac{\delta^2}{6}f_{xxx}. \tag{12}$$

Lagrange's derivation kept only the leading-order (ε and δ independent) terms. Here, we shall keep terms that are $\mathcal{O}(1)$ in these small quantities. To this end, the dimensionless kinematic boundary condition (3) becomes $\phi_y = \delta w_t + \delta\varepsilon\phi_x w_x + \mathcal{O}(3)$. Upon substitution of (11) and (12) and cancellation of a common factor δ, this formula can be rearranged to read

$$0 = w_t + \varepsilon w_x f + (1+\varepsilon w)f_x - (\delta/6)f_{xxx} + \mathcal{O}(2). \tag{13}$$

Similarly, differentiation of the dynamic boundary condition (4) with respect to x gives $0 = \phi_{xt} + (1/2)\varepsilon[(\phi_x)^2 + (\phi_y)^2/\delta]_x + w_x$. Again, expressions for ϕ_x and ϕ_y can be substituted from (11) and (12), giving

$$0 = f_t - (1/2)\delta f_{xxt} - \varepsilon f f_x + w_x + \mathcal{O}(2). \tag{14}$$

Now, equations (13) and (14) represent a pair of simultaneous equations for the unknowns $w(x,t)$ and $f(x,t)$. A careful attempt to solve these equations at zeroth and first order reveals a suitable relationship between f and w, viz.,

$$f = w - (\varepsilon/4)w^2 + (\delta/3)w_{xx} + \mathcal{O}(2).$$

Substitution of this expression into *either* (13) or (14) leads to the *same* equation up to first-order terms:

$$w_t + w_x + \varepsilon(3/2)ww_x + (1/6)\delta w_{xxx} = 0. \tag{15}$$

A further scaling of time and w allows us to effectively set $\varepsilon = \delta = 1$, which then leads to the usual dimensionless version of the KdV equation.

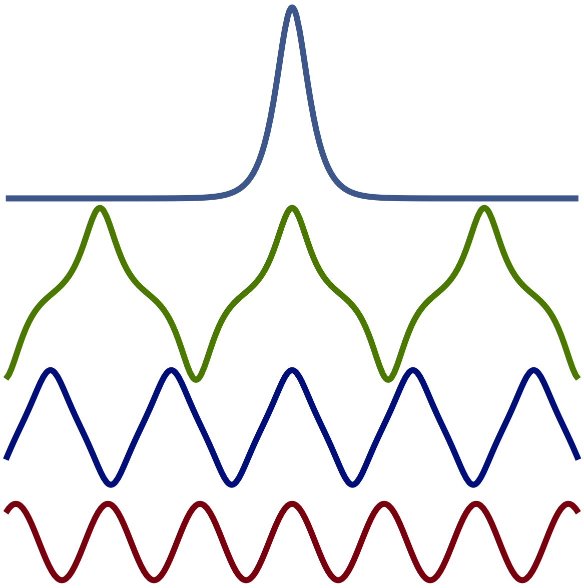

Figure 2: Cnoidal waves of increasing steepness and their infinite wavelength limit, the sech^2-solitary wave.

To me at least, this process of deriving (15) is somewhat involved and non-trivial, given the ubiquity of the KdV equation. In fact, there are many other contexts in which the KdV equation arises as a model of physical phenomena. It's just that it was first derived in the context of water waves. The whole field of soliton theory and integrable systems owes its origin to Martin Kruskal and Norm Zabusky's discovery in the 1960s that the KdV equation has a remarkable property of complete integrability. According to a popular search engine there are now over 200 scientific papers a year with Korteweg-de Vries or KdV in the title alone (let alone in the body of the text). I cannot do justice here to the myriad of properties of the solutions of this remarkable little partial differential equation (PDE) with just one nonlinear term, nor the many other integral PDEs that are its close cousin. Also many of these properties are not of direct relevance to our theme of water waves. In fact, for now, it will suffice for us to know that the KdV equation supports long, steep waves, known as cnoidal waves, which can be expressed as Jacobi elliptic functions. Cnoidal waves are periodic waves with arbitrarily long periods whose crests are more localised than sine waves (see Figure 2). In the limit as the wavelength tends to infinity we get a completely localised solitary wave, the famous KdV soliton whose profile is proportional to sech$(x - ct)$, where c is the wave speed. Note that for this large-amplitude wave the group velocity and phase velocity are identical.

Darrigol [3] provides the detailed history leading to the discovery of the KdV equation – from John Scott Russell's first canal-side observations in the early 1800s of a solitary wave, ending with Diederik Korteweg and his student Gustav de Vries's derivation of (15) in the latter's PhD thesis in 1895. Part of that story will feature in a future Westward Ho!, but here I want to provide just one remark. The KdV equation would appear to be an example of Stigler's law of eponymy:

> No scientific discovery is ever named after its original discoverer. [4]

Stigler himself attributes the discovery of this law to Robert Merton, meaning that the law applies to itself. Specifically, according to Darrigol's careful reading of the 19th century literature it was Joseph Boussinesq who first derived the KdV equation some 19 years before Korteweg and de Vries. Boussinesq went on to have a distinguished career, and his name abounds across fluid mechanics. In contrast, de Vries left science after his PhD and Korteweg did not publish further scientific papers of any great significance. Nevertheless, such is the impact of soliton theory that the names of Korteweg and de Vries are immortalised.

I said that waves that surfers seek are caused by wind. This is not entirely the case. I could hardly provide an article on surfing waves in the West Country without describing our region's regular, predictable, record-breaking surfable wave: the Severn bore (below). This is a rather different kind of solitary wave that occurs due to the high tidal range of the Severn (the second highest of any estuary in the world) and the abrupt narrowing of the estuary between Bristol and Gloucester.

Surfers on the Severn bore.

My late colleague Howell Peregrine was a worldwide authority on water waves, and spending his entire career in Bristol, he became an expert on the bore. He was a deeply kind man, but one who could carry on for hours and hours about the intricate details of a particular form of water wave, and would regularly organise trips to observe particularly good examples of the bore. As well as a consummate theoretician, he was known for taking quite remarkable photographs of naturally occurring wave phenomena including the bore; his photographs appear in many textbooks on fluid mechanics. The bore is not like the KdV soliton in that it is really a shock wave, a travelling front that separates two different flow states; calm and approximately laminar upstream, deeply turbulent downstream. Depending on the phase of the moon and various weather effects, the wave can be up to 2 m high.

It has become extremely popular to surf the bore. Why? Because, unlike an onshore breaker, a skilful surfer can follow the wave for mile after mile after mile. In March 2006, Steve King, a railway engineer and father of three from Gloucestershire, set a world record for the longest continous surf, riding the Severn bore for a distance of 7.6 miles (12.2 km), which he increased to 9.25 miles a few years later. He became something of a specialist and at the age of 48 braved a river full of crocodiles to set a new world record by riding Bono tidal wave on the Kampar River in Sumatra, Indonesia, even performing a handstand in the process. This record was recently beaten by an Australian, James Cotton, travelling 10.6 miles on the same river in March 2016.

So, what have we learned? Even though I have yet to learn to surf, I have tried to convey here just a small amount of the fascinating maths and mechanics inspired by water waves. I hope that I have not attracted the epithet that another colleague of mine once gave to a local who accosted him in a riverside car park early one morning as they waited for what turned out to be a rather pathetic wave; the Severn bore!

References

1 Guiney, J. (2016) Why do waves always break in odd-numbered groups? *Scientific American*, 27 Feb 2016. Available at: `http://tinyurl.com/Guiney2016`.

2 Constantin, A. (2009) On the relevance of soliton theory to tsunami modelling, *Wave Motion*, vol. 46, pp. 420–426.

3 Darrigol, O. (2005) *Worlds of Flow: A History of Hydrodynamics from the Bernoullis to Prandtl*, Oxford University Press, Oxford.

4 `en.wikipedia.org/wiki/Stigler's_law_of_eponymy`.