Westward Ho! Musing on Mathematics and Mechanics

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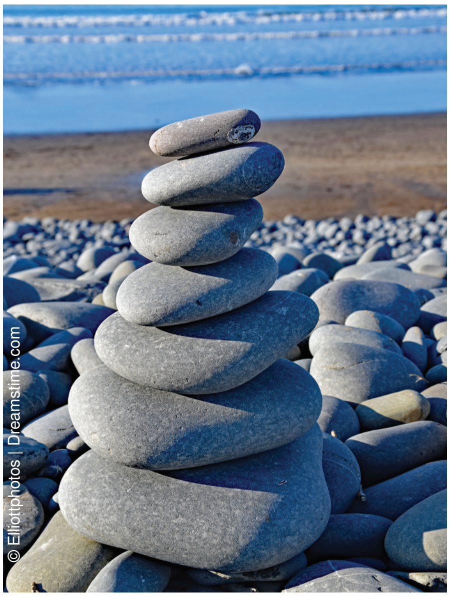


In the latest feature from Bristol and the West Country, Alan Champneys considers the life and work of the somewhat elusive 20th century mathematician G.N. Watson (1886–1965), probably the most important figure to be born in the unusual resort of Westward Ho!

Watson will be known to many generations of students through his influential textbook on mathematical analysis co-authored with E.T. Whittaker. A punctilious man, a computer before the age of modern computers, he was the first curator of Ramanujan's notebooks including the final 'lost' one. Watson's legacy is most keenly felt at the University of Birmingham, where he spent most of his career. That legacy is viewed here through the lens of another, less famous, Birmingham mathematician, David Wishart (1928–2003), a polymath, raconteur and amateur printer.

We shall also consider Watson's most famous result, a much-used lemma that bears his name, and in so doing ponder on the nature and importance of such 'elementary' mathematics.

'Elementary my dear Watson'

The Devon coastal resort of Westward Ho! is noted for its unusual name, a name that actually arises from a marketing ploy. It is named after the title of Charles Kingsley's 1855 novel [1] which is set in a fictional ocean facing quay near the town of Bideford. The book was a blockbuster and the 5th Earl of Portsmouth, with the unlikely name of Isaac Newton Wallop, saw the opportunity to cash in. In 1863 he formed the Northam Burrows Hotel and Villa Building Company and sought investors with the express purpose of opening a 'family hotel':

> The want of such accommodation has long been felt, and as no attempt to supply it has hitherto been made by individuals .... The salubrity and beauty of the North of Devon have long been known and appreciated ... the recent publication of Professor Kingsley's 'Westward Ho' [sic] has excited increased public attention to the western part, more especially, of this romantic and beautiful coast. Nothing but a want of accommodation for visitors has hitherto prevented its being the resort of families seeking the advantages of sea bathing, combined with the invigorating breezes of the Atlantic ... [2].

The foundation stone was laid in February 1864 and The Royal Hotel was completed in time for the 1865 season. Charles Kingsley himself was invited to come and open it. At first he declined, believing that the development of the area would ruin its rugged beauty, so a quick change of name was undertaken. The hotel became The Westward Ho! Hotel. Charles Kingsley attended, took his cheque and it is said never came near the place again. The hotel became a success and villas were erected in the 75 acre estate surrounding it. As further development took place, the expanding settlement soon acquired the name of Westward Ho!

The Royal North Devon Golf club, which opened the same year as the hotel, is the oldest in England and Wales. Then, in 1874, the United Services College was opened nearby. It boasts Rudyard Kipling among its famous former pupils, and his story collection *Stalky & Co* is based on his experiences there. Staff for the school, as well as for hotel, golf course and other attractions soon brought permanent residents – they would have otherwise found it hard to commute via the steep track from Bideford. Holy Trinity Church was built in 1869, soon followed by a Baptist church and several pubs. More growth in the tourist business came with the coming of a railway line to Bideford in 1901. Westward Ho! gained the status of a separate village with all the facilities one might expect.

The exclamation mark is an intentional part of the village name. It is the only official place name in the British Isles to contain such a punctuation mark. However it is not a world record. That goes to a small Canadian hamlet which has, not one, but two (!!) exclamation marks as part of its name. Saint-Louis-du-Ha! Ha!, just South of the St Lawrence estuary in Quebec is thought to get its name from the haha – from an archaic French word for obstacle – formed by the nearby lake Témiscouata which is a major obstacle to local travel.

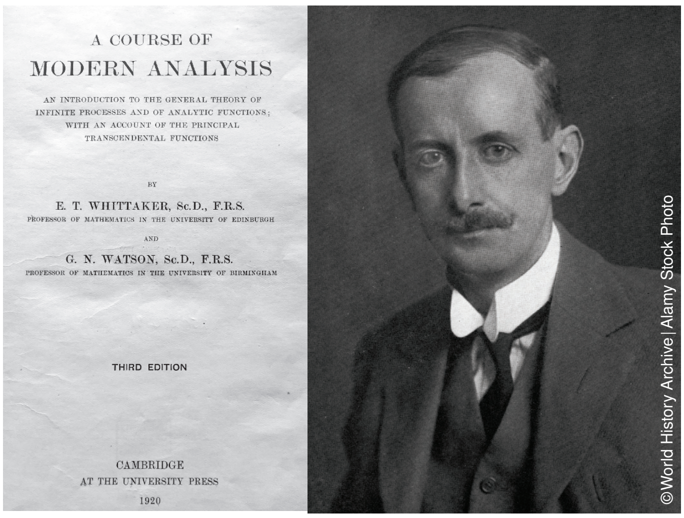
Westward Ho!'s most famous mathematical son, George Neville Watson was born on 31 January 1886 in the village. He should not be confused with British number theorist G.L. Watson (1909–1988) or Australian statistician G.S. Watson (1921–1998) both of whom made strong contributions to 20th century mathematics. At the time of his birth G.N.'s father, George Wentworth Watson, was a teacher at the United Services School. G.W. Watson would later go on to play a large role in the publication of *The Complete Peerage*, a 13-volume encyclopedia of the British peerage, generally accepted as the greatest British achievement in the field of genealogy from the pre-digital age.

Neville Watson, as the younger George Watson came to be known, was educated at St Paul's School from where he won a scholarship to study maths at Trinity College Cambridge in 1904. He was Senior Wrangler in 1907, and gained a fellowship at Trinity in 1910. After a brief spell as an assistant professor at University College London for the duration of the First World War, Watson was appointed in 1918 as the Mason Professor of Mathematics at the University of Birmingham. He was elected a Fellow of the Royal Society in 1919. In 1925 he married Elfrida Gwenfil Lane, they set up home south of Birmingham and had one son, also called George. Watson remained at the University of Birmingham for the rest of his career, taking retirement in 1951.

Figure 1: Modern day Westward Ho! (left) and in 1899 (right) with the hotel in the foreground.

Generations of students, from the 1920s up to the beginning of 21st century, will have become acquainted with G.N. Watson through the mathematical analysis textbook that bears his name, together with that of his collaborator and erstwhile teacher Edmund Taylor Whittaker. Published in 1915, *'A course on Mathematical Analysis: an introduction to the theory of infinite processes and analytic functions: with an account of the principal transcendental functions'* [3], to give it its full title, E.T. Whittaker and G.N. Watson was actually a much expanded second edition of a 1902 first edition in Whittaker's name alone. Third and fourth editions were published in 1920 and 1927, with numerous re-printings. In fact it remains in print today.

The book is completely rigorous, in keeping with the traditions of Cambridge analysis. Yet, for its time, it takes a more modern pedagogical view, written primarily as a study aid for the diligent reader. It has numerous exercises – many taken from the world famous Cambridge Mathematical Tripos. The book's influence also goes well beyond the English speaking world, with evidence to suggest it was a prime study aid of the French Bourbaki movement [4].


A COURSE OF

MODERN ANALYSIS

AN INTRODUCTION TO THE GENERAL THEORY OF INFINITE PROCESSES AND OF ANALYTIC FUNCTIONS; WITH AN ACCOUNT OF THE PRINCIPAL TRANSCENDENTAL FUNCTIONS

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THIRD EDITION

CAMBRIDGE
AT THE UNIVERSITY PRESS
1920

Figure 2: G.N. Watson and the front page of his textbook with E.T. Whittaker

Whittaker and Watson met at Trinity College, Cambridge; although within two years of the latter's arrival as an undergraduate Whittaker left to become Astronomer Royal in Ireland and then to take up a chair in Edinburgh. Nevertheless the two became lifelong friends and correspondents. Not only did they share mathematical interests, but it would seem they had similar personalities. Both were meticulous – what we might call 'details men' in modern parlance – and humble, with a generous spirit, wishing to make a contribution to the world rather than seek glory for themselves. They were throwbacks to an earlier era, sharing a dislike for telephones, a love of public transport, with a consequent hatred of cars, and a strong interest in the history of their subject. Both remained devoted to Trinity throughout their lives, with the walls of the Watson family home being adorned with a collection of prints of the college and famous Trinity men.

Whittaker & Watson [3] was followed by a sole authored book, considered by Watson himself to be his greatest achievement, *A Treatise on the Theory of Bessel Functions* (1922) [5]. The book has two stated aims; first, to introduce students to practical applications of complex variable theory and, second, to compile everything that is known about the theory of Bessel functions and their application across mathematics and physics. A remarkable feature is that, after more than 600 pages of detailed mathematical argument, an appendix contains almost 90 pages of tabulated numerical values of various different Bessel functions, which Watson calculated by hand. Watson was a master calculator, his primary expertise being in the theory and practical application of asymptotic expansions.

To the modern mathematician, it is easy to forget the importance of the families of 'special' mathematical functions such as the Bessel functions. In the pre-computer age, the aim of the applied mathematician was often to approximate and reduce problems to integrals or differential equations that can be expressed in certain canonical forms. The solution of each different form is then expressed in terms of these special functions. For example, the order-α Bessel function of the first kind $J_\alpha(x)$ can be expressed as the solution of the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\alpha^2)y=0,$$

or, for integer values α, via the integral

$$J_\alpha(x)=\frac{1}{\pi}\int_0^\pi \cos(n\tau-x\sin(\tau))\,\mathrm{d}\tau.$$

These functions, which broadly speaking behave like exponentially modulated sine waves (see Figure 3), do not possess closed-form expressions in terms of elementary transcendental functions. Therefore, any practising applied mathematician would need to be familiar with their properties, just as we are these days with sines and cosines, and would need to refer to the tables whenever a numerical value was required.

Watson's book remained the definitive source of reference on Bessel functions for more than 50 years, until the rise of the NIST handbook [6] and modern computer algebra systems.

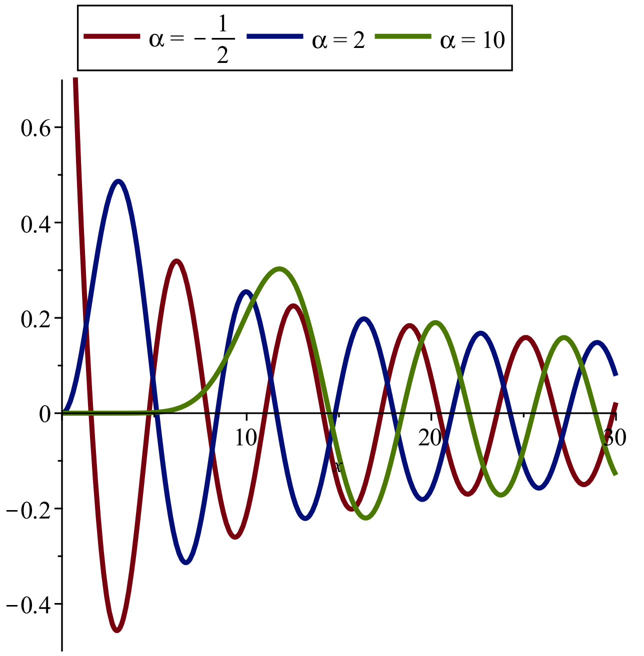

Figure 3: Graphs of Bessel functions J_α for three different values of α.

The result for which G.N. Watson is undoubtedly most well known is a method for approximating certain exponential integrals. The so-called *Watson's lemma* concerns integrals of the kind that often arise when taking Laplace transforms:

$$F(s) = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t,$$

in the limit of large s. The result assumes that the function $f(t)$ can be written in the form

$$f(t) = t^\alpha g(t), \qquad \text{for some real } \alpha > -1,$$

where

$$g(t) = \sum_{k=0}^{\infty} a_k t^k$$

is an analytic function that is bounded as $t \to \infty$ by e^{at} for some finite a. The result states that

$$F(s) \sim \sum_{k=0}^{n} a_k \frac{\Gamma(\alpha+k+1)}{s^{\alpha+k+1}} + \mathcal{O}\left(\frac{1}{s^{\alpha+n+2}}\right) \quad \text{as} \quad s \to \infty. \tag{1}$$

Here $\Gamma(x)$ is the well-known gamma function, which is equal to the factorial for integer arguments – $\Gamma(n) = n!$ – and can be defined as the analytical continuation of the expression

$$\Gamma(x) = \int_0^\infty \xi^{x-1}\,\mathrm{e}^{-\xi}\,\mathrm{d}\xi, \tag{2}$$

that is valid for Re$(x) > 0$.

At a formal level, the formula (1) is straightforward to derive. Simply expand f as a power series and then perform a change of variables $t = \xi/s$ so that each term leads to an integral of the form (2). The less trivial part of the proof, although still elementary, is to show that this formula converges, and that there is no nasty remainder term that blows up as t or $s \to \infty$. This is achieved by splitting the domain of integration into a finite piece $[0, T]$ and the remainder $[T, \infty)$. On the latter piece, we show that the exponential bound of $f(t)$ will be quenched by the factor e^{-st} for sufficiently large s. On the finite piece we show that the dominant term arising from each term in the expansion of f is given by the gamma-function expression, and then we use the fact that g is analytic to give an estimate of the remainder term.

Watson's lemma appeared as a small result in a paper of his that attempts to estimate certain properties of parabolic cylinder functions. It has proved extremely useful though; for example in some of the calculations that result from using the method of steepest descent, inverting Laplace transforms, and in the WKB (Wentzel–Kramers–Brillouin) method that is widely adopted in acoustics and semi-classical quantum mechanics for estimating properties of waves with slowly varying parameters.

For the ten years leading up the Second World War, Watson devoted his attention to the unpublished work of Srinivasa Ramanujan (1887–1920), the largely self-taught mathematical genius who emerged from poverty in India. The story of the remarkable collaboration between Ramanujan and G.H. Hardy has been told many times over. What is perhaps less well known is Watson's pivotal role in bringing the genius's results to wide attention.

After Ramanujan's untimely death in 1920, his widow donated all his papers and notebooks to the University of Madras. In correspondence, Hardy persuaded the University to send them to the UK in an attempt to get the work published. In turn, the papers were sent to Watson who, together with B.M. Wilson from the University of Liverpool, was assigned the task of reworking, checking, providing proofs or references to the literature, tidying up and ultimately publishing the results.

Figure 4: Srinivasa Ramanujan

The task was monumental. Ramanujan would work on a slate and then transcribe only the most important formulae by hand to a notebook or sheet of paper. Most of his unpublished work was preserved in a series of three dense notebooks. These contained terse summaries of his results from 1903 to 1914, which he had recorded in order to try to persuade benefactors to support him. It is estimated that the notebooks contain about 4000 theorems, usually stated without proof. While containing some rediscoveries of known results, most of the work is truly original, showcasing both Ramanujan's remarkable creativity as well as his savant-like calculation abilities.

Watson and Wilson had intended to write a single monograph, a book of 600 pages or so, to at least do some justice to the material. In the event, the project did not come to fruition, partly due to the premature death of Wilson in 1935 and probably because

Watson became overwhelmed with the enormity of the project. Nevertheless, Watson appears to have expended a great deal of effort. For example, in the days before photocopiers, he made a transcript of Ramanujan's notebooks in '288 closely written foolscap pages' [7]. This manuscript did not see the light of day during Watson's lifetime, although Watson did publish many original mathematical papers during this period that contained results obtained through reworking of Ramanujan's notes, or inspired by them.

In 1957 the Tata Institute in India published simple facsimile copies of Ramanujan's original manuscripts. But it was not until the 1980s that publication of fully explained and reworked versions of Ramanujan's notebooks began to be published, through the monumental efforts of Bruce C. Berndt from the University of Illinois. Berndt's labours appear in a massive five-volume work of more than 2500 pages, which appeared between 1985 and 1998 [8]. It is no wonder that Watson became overwhelmed with the task.

In fact, Watson's obsession with detail seems to have led to him finding it difficult to be a completer-finisher in the later years of his life. He published little after the Second World War. Upon his death, Robert Rankin of the University of Glasgow, who had been Watson's immediate successor as the Mason Professor in Birmingham, was initially given the task of organising Watson's unpublished work. To his horror, Watson's widow showed him a room in their house, devoid of furniture, knee deep in unpublished manuscripts. These included a massive, but incomplete further edition of the Whittaker and Watson textbook, a monograph on *Three Decades of Midland Railway Locomotives* and a great deal of material relating to Ramanujan (see preface to [9]). The task of further editing Watson's papers was given to E.T. Whittaker's son, J.M. Whittaker, who was appointed by the Royal Society to write Watson's obituary in its *Biographical Memoirs* [7].

Among the Ramanujan documents in Watson's possession was some less well organised material, in the form of a series of loose sheets from the last year of Ramanujan's life after he had returned to India in poor health. During this time, despite being mostly confined to bed, he had continued to work, and had written to Hardy concerning further results, including remarkable properties of the so-called mock theta functions. This work became known as *Ramanujan's Lost Notebook* and it is widely held to contain the genius's deepest and most profound results.

Watson's Ramanujan material was donated to Trinity College Library. Then in 1976, George Andrews from the University of Pennsylvania, a Ramanujan enthusiast, took a sabbatical in Cambridge with the express desire to go through all of the papers in the Watson archive. It was after this visit that he announced to the world that he had found the lost notebook (although really just a collection of sheets). The reworking of that material has now appeared in four further volumes co-authored by Andrews and Berndt, with the final one only being published as recently as 2013 [9].

Watson was very much a lone scholar. Unlike common practice today, he did not regard mathematical research as a team sport. Apart from his book with Whittaker, he only ever published one joint paper. He had no known PhD students, nor did he establish any kind of school around him. Nevertheless, he did believe in service to the university and to other mathematicians. He was active in his support for the London Mathematical Society, serving as secretary from 1919 to 1933, president from 1933 to 1935 as editor of their *Proceedings* until 1946. The society awarded him their De Morgan Medal in 1947. Also,

> he was the university's expert on the timetable; students with unusual combinations of subjects usually had to be referred to him for advice, and for many years after his retirement the dates of the academic year were governed by the 'Watsonian cycle' ... He took great trouble with the style of his letters and his conversation and enjoyed finding a pungent phrase to express his points of view or his criticism ... he made no secret of his aversion to cars, telephones, and fountain pens. He loved trains – whose timetables were as familiar to him as those of the university lectures – and unusual stamps [10].

I first heard of Watson because the University of Birmingham Mathematics Building is named after him. As an undergraduate I remember lectures by David Wishart, a statistician, applied mathematician and sometime raconteur who taught the final-year option on the history of mathematics. I did not take those lectures, but I did take a lecture course with him on nonlinear systems and asymptotic methods. This was a new module that had been added to the option list by Prof Tony Skyrme who had sadly died the previous summer (the remarkable Skyrme will doubtless feature in a future Westward Ho! piece in this magazine). Wishart was given the job of trying to do justice to Skyrme's ideas. It is said that Wishart would lecture in parentheses. He would start on a topic, begin a digression, which would turn into a historical anecdote, then a related piece of mathematics, another anecdote, and so on. Yet, if each digression was the opening of a bracket, then each bracket would eventually be closed, in the right order, so that by the end of the lecture (often too close to the end for those who had to prepare for an exam in the subject) he would return to the intended topic of the day's class.

I recall Wishart telling us of the quiet, elegant, private man who had frequented the maths department at Birmingham when Wishart was a young lecturer. Although having long since formally retired, G.N. Watson would still regularly attend. He would travel in by bus from his Warwickshire home, wearing tweeds and hat. Even in the late 1950s Wishart remarked that Watson seemed like a throwback to an earlier era.

Wishart himself would probably not have survived in our modern target-obsessed university departments. As his colourful obituary shows [11], he did not appear to make a distinction between his home and professional life. At the end of each lecture course, he would invite students to his home for a buffet lunch. He became chairman of the Birmingham Community Health Council and of the Birmingham branch of the Consumers' Association. He was a keen member of the Birmingham Chamber Music Society, in charge of producing and printing its concert programmes. He was a keen book collector and chairman of the Birmingham Bibliographical Society.

He was deeply involved in the Royal Statistical Society, for many years being editor of one of its journals. The typesetting of the journal seemed to pique his interest in mathematical printing and in the history of printing and typeface more generally, especially of unusual non-European languages. He took possession of a collection of seven Victorian printing machines, and had a collection of more than 100 different typefaces and an array of ornaments and blocks purchased from Midland printing firms as they moved away from the outdated technology.

I seem to recall that he and his family had bought the house next door, so that they owned two large back-to-back semi-detached properties. The main part of the adjoining property was rented out to students, but there was a large outbuilding in which he started his own private printing press. The so-called Hayloft Press (not to be confused with other companies bearing the same name), published rare and unusual books ranging from a bicentennial reprint of Tom Paine's short essay, 'The Last Crisis', to a fully illustrated and bound copy of *A Shropshire Lad* by A.E. Housman. Each of them was printed in editions of less than 300, but achieved a wide distribution and a diligent readership. He would also produce numerous items of stationery, greeting cards and miscellaneous ephemera containing pithy or inspiring quotes. The archive of this remarkable enterprise is now held at the University of Manchester [12].

During one lecture with him, Wishart announced to the class that he had bought from Oxford University Press their entire Egyptian hieroglyphic typeface collection, which they were jettisoning after they had moved to digital printing. The collection would be arriving by van the next day and he asked for volunteers to help unload them. I recall a moderate-sized furniture lorry arriving full, almost to the brim, of heavy wooden trays, each full of metal type pieces. These had to be transported by hand to the 'Hayloft'. We were each rewarded with lunch and a personalised printed greeting card. Mine, printed on beautiful purple card was a quote from Mark Kac:

> A demonstration is a way to convince a reasonable man, and a proof is a way to convince a stubborn one.

I kept this card for many years, although sadly it got lost quite recently along with other posters, cards and memorabilia in one of my many office moves.

I have often pondered on this quote. It tends to summarise my own philosophy as an applied mathematician (although I would prefer a version without a male-specific pronoun). I am a reasonable person. Don't misunderstand me. There is a lot to be said for scientific stubbornness; for tenacity, for not taking things at face value, to seek one's own understanding and to convince oneself of the veracity of a 'known' result. But there comes a point at which nailing every last epsilon and delta seems like an overly pedantic exercise. In my view, the number of results that are undoubtedly true greatly outnumber those that can be rigorously proved, and this is even more true nowadays since we can compute approximate numerical solutions to many problems in the blink of an eye (or the flash of a photon).

I don't suppose that G.N. Watson would have approved of Kac's categorisation of reasonableness. Watson was from the tradition of analysis in which mathematical rigour is valued above all else. Therefore I feel sorry for him that his most famous result, Watson's lemma, is essentially just an elementary calculation. But then, in my view, often what may seem like the most simple and elementary mathematical result, can turn out to be the most important and far-reaching, and to have the biggest influence.

For me, breakthroughs, if I ever have them, come not from complexification, nor from the scaling of seemingly unreachable heights. Rather they come from finally, after wandering stupidly in the fog of confusion, getting a sudden glimpse at simplicity. How many times when we finally solve a problem do we put our heads in our hands and say 'Oh, of course, it's obvious!'?

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