Imprecise statistical inference for accelerated life testing data: imprecision related to the likelihood ratio test

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Abstract In this paper, we present a new imprecise statistical inference method for accelerated life testing data, where nonparametric predictive inferences at normal stress levels are integrated with a parametric Arrhenius-Weibull model. The method includes imprecision which provides robustness with regard to the model assumptions. The imprecision leads to observations at increased stress levels being transformed into interval-valued observations at the normal stress level, where the width of an interval is larger for observations from higher stress levels. Simulation studies are presented to investigate the performance of the proposed method.

1. Introduction

Accelerated life testing (ALT) has frequently been used to obtain information on the lifespan of devices. Testing items under normal conditions can require a great deal of time and expense. To determine the reliability of devices in a shorter period of time, and with lower costs, ALT is frequently used. In ALT, a unit is tested under levels of physical stress (e.g. temperature, voltage, or pressure) greater than normal operating conditions. Using this method, devices will fail more quickly, enabling estimation of the lifetime under normal conditions via extrapolations using an ALT model. There is a wide range of designs for ALT, including constant-, step- and progressive-stress testing [Nelson 1990]. Two components usually used to extrapolate a device's lifespan under normal use conditions from ALT results are the life time distribution (e.g. Weibull or log-normal) and the relationship between failure time and stress level (e.g. Arrhenius law or power law).

In this paper, we consider the Arrhenius model and a Weibull lifetime distribution for a constant-stress ALT. The Arrhenius model is based on a physical/chemical theory, and thus is often an appropriate model to use when the failure mechanism is driven by temperature [Nelson 1990]. The Weibull distribution is usually used for examining component, system or product life. The Arrhenius-Weibull model is adapted to the current research on imprecise statistical approaches to establish the use of imprecision in modelling the nature of the relationship between stress levels and unit failure rates. The main issue here is how to extrapolate the failure data from units tested at higher-than-normal stress levels to units operating at normal level.

To explain the Arrhenius-Weibull model assumptions, we must understand that it is based on differential failure times at varying stress levels. For example, K_0 represents the actual stress level, and there are $m \geq 1$ increased stress levels, with stress K_i at level $i \in \{1,...,m\}$. We parameterize the Weibull distribution such that its survival function is

$$P(T > t) = \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right],$$

with α the scale parameter and β the shape parameter. The Weibull distributions for different stress levels are assumed to have different scale parameters, α_i for level i, but the same shape parameter β. Therefore, the link function of scale parameter α_i should be used for establishing a connection between different stress levels i, and is assumed to satisfy the function

$$\alpha_i = \alpha_0 \exp\left(\gamma/K_i - \gamma/K_0\right). \tag{1}$$

Where using this model with temperature as stress levels, K_0 is the normal temperature (Kelvin) at stress level 0 , K_i is the higher temperature (Kelvin) at stress levels i, and γ is the parameter of the Arrhenius link function model. So there are in total three parameters that need to be estimated, α_0, β, and γ.

In this work, we will apply the pairwise likelihood ratio test to investigate equality of different scale parameters, with shape parameter being equal for all stress levels, as explained in Section 2 [Nelson 1990, Alam and Sudhir 2015]. In this paper, we assume the same β for all levels, so difference between stress levels is only modelled through the α_i, and hence in the model through the γ parameter of the link function between different stress levels. We develop the use of likelihood ratio test to obtain the interval of values of the γ parameter of the link function for which the null hypothesis that two groups of failure data come from the same underlying distribution, is not rejected. In this paper, we assume that there are no right-censored observations. Note further that at the normal stress level 0 we assume to have failure observations. We briefly comment on this in the last section.

In this paper, we develop a new likelihood ratio test based method for analysing ALT data based on imprecise probabilities, where nonparametric predictive inferences (NPI) at a normal stress level are integrated with a parametric Arrhenius-Weibull model. This new method consist of two steps. We assume Arrhenius-Weibull as a complete model for all levels in the first step. Then we derive an interval for γ values of the Arrhenius link function model by pairwise likelihood ratio tests, which allows observations of increased stress levels to be transformed to interval-valued observations at the normal stress level in the second step. Then we use NPI at the normal stress level for predictive inference on the failure time of a future unit operating under normal stress.

Nonparametric Predictive Inference (NPI) is a statistical method, which provides lower and upper survival functions for a future observation based on past data using imprecise probability [Augustin, et al 2014, Coolen 2011]. [Hill 1968] proposed an assumption which gives direct conditional probabilities for a future random quantity which depend on the values of related random qualities [Augustin and Coolen 2004, Coolen 2006]. Given ordered observations $x_1 < x_2 < ... < x_n$, and defining $x_0 = 0$ and $x_{n+1} = \infty$, the NPI lower and upper survival functions for a future observation X_{n+1} are

$$\underline{S}_{X_{n+1}}(t) = \frac{n-j}{n+1}, \text{ for } x \in (x_j, x_{j+1}), \ j = 0, ..., n. \tag{2}$$

$$\overline{S}_{X_{n+1}}(t) = \frac{n+1-j}{n+1}, \text{ for } x \in (x_j, x_{j+1}), \ j = 0, ..., n. \tag{3}$$

The imprecision, the difference between the upper and lower survival functions, reflects the amount of information in the data.

This paper is organized as follows: in Section 2, the novel method of imprecise predictive inference is introduced based on ALT failure data and pairwise likelihood ratio tests. In Section 3 we illustrate our proposed method in an example. Section 4 presents results of initial simulation studies that investigate the performance of the proposed method, where the data are simulated using the assumed Arrhenius-Weibull model. Section 5 presents concluding remarks.

2. Predictive inference based on ALT data and pairwise likelihood ratio test

The aim of this section is to combine NPI with the fully parametric model used in our new statistical method analysing data from ALT. The use of NPI here provides lower and upper survival functions for a future observation at the normal stress level, based on all failure data. This new statistical method for data in ALT divides neatly into two steps. First, the basic Arrhenius-Weibull model is adopted [Nelson 1990], and the pairwise likelihood ratio test is used between the stress levels K_i and stress level K_0, to obtain the intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ for the parameter γ for which we do not reject the null hypothesis that the data transformed from stress level i to normal stress level 0, and the original data obtained at the normal stress level 0, come from the same Weibull distribution, where $i = 1, ..., m$. With these m pairs $[\underline{\gamma}_i, \overline{\gamma}_i]$, we define $\underline{\gamma} = \min \underline{\gamma}_i$ and $\overline{\gamma} = \max \overline{\gamma}_i$.

The Arrhenius link function for scale parameters α_i should be identified to establish a connection between the different stress levels i. Regarding to this link function model, an observation t^i at the stress level i, subject to stress K_i, can be transformed to stress level 0, with the transformed observation from level i to level zero represented by the equation

$$t^{i \to 0} = t^i \exp\left(\frac{\gamma}{K_0} - \frac{\gamma}{K_i}\right). \tag{4}$$

In the second step, we apply the data transformation using $\underline{\gamma}$ for all $i = 1, ..., m$ to get transformed data at level 0 which leads to NPI lower survival function $\underline{S}$. Similarly, we apply the data transformation using $\overline{\gamma}$ for all $i = 1, ..., m$ to get transformed data at level 0 which leads to NPI upper survival function $\overline{S}$. Note that each

Table 1: Failure times at three temperature levels

Case	$K_0 = 283$	$K_1 = 313$	$K_2 = 353$	$K_1 = 313$ (*1.4)	$K_2 = 353$ (*0.8)
1	2692.596	241.853	74.557	338.595	59.645
2	3208.336	759.562	94.983	1063.387	75.987
3	3324.788	769.321	138.003	1077.050	110.402
4	5218.419	832.807	180.090	1165.930	144.072
5	5417.057	867.770	180.670	1214.878	144.560
6	5759.910	1066.956	187.721	1493.739	150.176
7	6973.130	1185.382	200.828	1659.535	160.662
8	7690.554	1189.763	211.913	1665.668	169.531
9	8189.063	1401.084	233.529	1961.517	186.823
10	9847.477	1445.231	298.036	2023.323	238.429

Table 2: Accepted γ for likelihood ratio

Significance level	0.99		0.95		0.90	
Stress level	$\underline{\gamma}_i$	$\overline{\gamma}_i$	$\underline{\gamma}_i$	$\overline{\gamma}_i$	$\underline{\gamma}_i$	$\overline{\gamma}_i$
Case 1: K_1 K_0	4060.018	6605.752	4424.881	6261.168	4593.700	6100.653
K_2 K_0	4377.043	5602.321	4550.205	5434.908	4630.511	5357.037
Case 2: $K_1 * (1.4)$, K_0	3066.539	5612.273	3431.402	5267.689	3600.221	5107.174
$K_2 * (0.8)$, K_0	4695.498	5920.775	4868.660	5753.363	4948.965	5675.492

observation at an increased stress level transforms into an interval-valued observation at the normal stress level 0, where the width of an interval is larger for an observation from a higher stress level.

Note that, if the model fits really well, we expect most $\underline{\gamma}_i$ values to be quite similar, as well as most $\overline{\gamma}_i$ values. If the model fits poorly, $\underline{\gamma}_i$ are most probably very different, or $\overline{\gamma}_i$ are very different, or both. The proposed method in Section 2 is illustrated by an example using simulated data in Section 3.

3. Example

In this section we present two cases. In case 1 we simulated data at all levels using the parametric model for the link function we assume for the analysis. In case 2 we change these data such that the assumed link function will not provide a good fit anymore and we show the change to the resulting interval $[\underline{\gamma}, \overline{\gamma}]$. We assumed the temperature stress levels to be $K_0 = 283$, $K_1 = 313$ and $K_2 = 353$ Kelvin. We generated ten observations from a Weibull distribution at each stress level integrated with the Arrhenius link function. The Weibull distribution at K_0 had $\beta = 3$ and $\alpha_0 = 7000$, and the Arrhenius parameter γ was set at 5200. This model keeps the same shape parameter at each temperature, and the scale parameters are linked by the Arrhenius relation, which led to $\alpha_1 = 1202.942$ at K_1 and $\alpha_2 = 183.0914$ at K_2. Ten units were tested at each temperature, so a total of 30 units is used in the study. The failure times are given in Table 1.

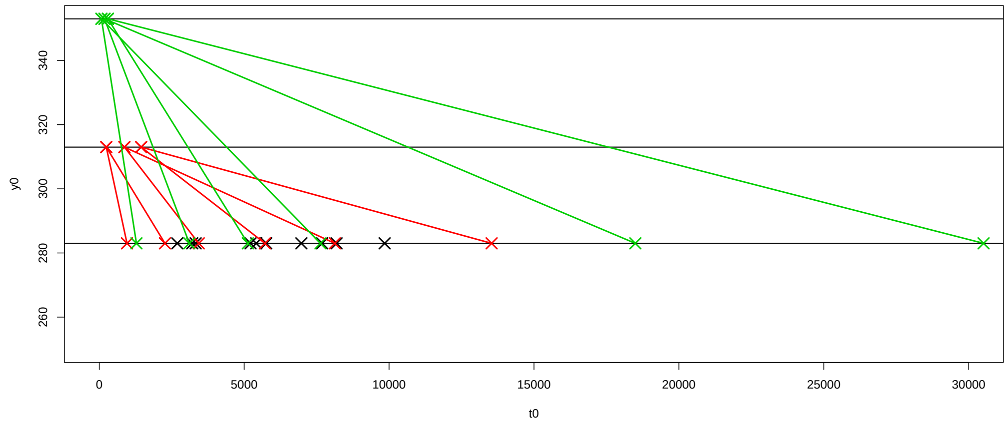

Figure1. Transformed data using $[4060.018, 6605.752]$

To illustrate the pairwise likelihood ratio tests method using these data, we assume the Weibul distribution at each stress level and the Arrhenius link function for the data. To obtain the accepted intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ of the values γ_i for which we do not reject the null hypothesis with regard to the well-mixed data transformation, we used the pairwise likelihood ratio test between K_1 and K_0 and between K_2 and K_0. The results of the accepted intervals $[\underline{\gamma}_i, \overline{\gamma}_i]$ for three significance levels are giving in Table 2.

Then, we transformed the data using the $[\underline{\gamma}, \overline{\gamma}]$ values. All observations at the increased stress levels were transformed to the normal stress level. Therefore, the observations at the increased stress levels K_1 and K_2 are transformed to interval-valued observations at the normal stress level K_0. We briefly illustrate this using only a small part of the data in Figure 1. We transformed the data from the higher stress levels K_1 and K_2 using $[4060.018, 6605.752]$ with 0.99 level of significance, mixed with the original samples at the normal stress level K_0. Note that, in Figure 1 the two largest transformed are actually the $\underline{\gamma}$ and $\overline{\gamma}$ transformations of the largest observation from level K_2. So, this illustrates the key property of our method, that data transformed from higher levels tend to be wider intervals at the normal level.

The NPI lower survival function can be obtained by taking from the pairwise stress level K_1 to K_0 or K_2 to K_0 always the min of the $\underline{\gamma}_i$ with significance levels 0.99, 0.95 and 0.90 values, giving in Table 2. Similarly, the NPI upper survival function can be obtained by taking from the pairwise stress level K_1 to K_0 or K_2 to K_0 always the max of the $\overline{\gamma}_i$ with levels of significance 0.99, 0.95 and 0.90 values, given in Table 2. In case 1, we take $\underline{\gamma} = 4060.018$ and the $\overline{\gamma} = 6605.752$, $\underline{\gamma} = 4424.881$ and $\overline{\gamma} = 6261.168$, and $\underline{\gamma} = 4593.700$ and $\overline{\gamma} = 6100.653$ of the pairwise K_1, K_0 with 0.99, 0.95 and 0.90 significance levels, respectively. We used all the above $\underline{\gamma}$ and $\overline{\gamma}$ values to transform the data to the normal stress level 0, see Figure 2(a).

In case 2, we multiply the data at level K_1 by 1.4 and in addition we multiply the data at level K_2 by 0.8. The simulated data values are given in the last two columns in Table 1. We take $\underline{\gamma} = 3066.539$ and $\overline{\gamma} = 5920.775$, $\underline{\gamma} = 3431.402$ and $\overline{\gamma} = 5753.363$, and $\underline{\gamma} = 3600.221$ and $\overline{\gamma} = 5675.492$ for 0.99, 0.95 and 0.90 significance levels, respectively. We used all the above $\underline{\gamma}$ and $\overline{\gamma}$ values to transform the data to the normal stress level 0, see Figure 2(b). In these figures, the lower survival function $\underline{S}$ is labeled as $\underline{S}$ $(\underline{\gamma}_i)$ and the upper survival function $\overline{S}$ is labeled as $\overline{S}$ $(\overline{\gamma}_i)$. These figures show that higher significance level leads to more imprecision for the NPI lower and upper survival functions. Note that in Case 2, the observations at level K_1 have increased, leading to smaller $\underline{\gamma}_1$ and $\overline{\gamma}_1$ values, and this leads to the lower and upper survival functions decreasing in comparison to Case 1. Also, the observations at K_2 stress level in case 2, have increased, resulting in larger values for $\underline{\gamma}_2$ and $\overline{\gamma}_2$, and this leads to the lower and upper survival functions increasing in comparison to Case 1. The combined effect, based on $\underline{\gamma}$ and $\overline{\gamma}$, is therefore less obvious.

4. Simulation studies

The predictive statistical method we propose based on imprecise probabilities for ALT using the pairwise likelihood ratio tests with the parametric Arrhenius-Weibull model, is intuitively attractive because it shows

(a) Case 1 (b) Case 2

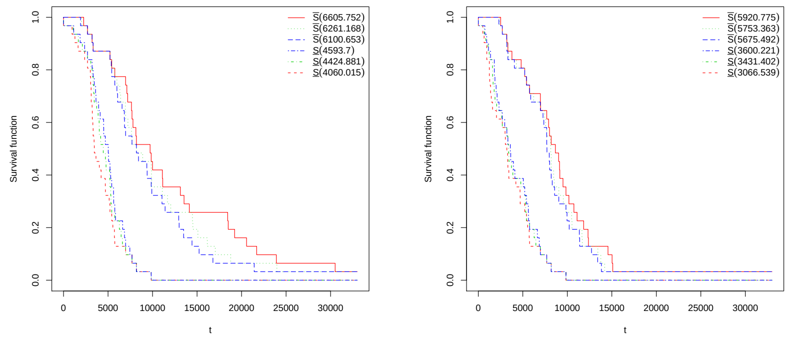

Figure2. The NPI lower and upper survival functions: Case 1 and Case 2.

some robustness, and demonstrates the effect of imprecision when we lack perfect information about the link between different stress levels. In this section, we apply simulation studies to investigate the performance of our proposed method when considering the assumed Arrhenius-Weibull model as the true underlying model.

To investigate the predictive performance inference of our new method for ALT data, we conduct a simulation study. We assumed the temperature stress levels to be $K_0 = 283$, $K_1 = 313$, and $K_2 = 353$. In this analysis, we run the simulation 10,000 times with the data simulated from the Weibull distribution and using the parameters shape parameter = 3, scale parameter = 7000, and the link function parameter γ = 2000, using the assumed Arrhenius link function model between different stress levels, where we have used $n = 10, 50, 100$ assigned to each stress level. We applied the method described in Section 2, with level of significance 0.95. For good performance of the method, we check the results by taking into account a future observation with actual data transformed to the normal stress level K_0 and how well the future observation mixes among them. We examined the performance in the quartiles of NPI lower and upper survival functions for $q = 0.25, 0.50, 0.75$, and whether or not the future observation at the normal stress level has exceeded these quartiles in the right proportion. One can use different quartiles but these quartiles provided a good indicator of the performance of our method. We conducted simulation with assumed Arrhenius-Weibull model, hence with data generated from the same model.

For good performance of the method, we require that the future observation for each run at the normal stress level has exceeded the first, second, and third quartiles of the NPI lower survival functions just over proportions $0.75, 0.50$ and 0.25 of all runs, respectively, and for the NPI upper survival functions just under proportions $0.75, 0.50$ and 0.25. Figure 3 presents the results of the performance of our method when the Arrhenius model is the assumed model for the simulated data and for the analysis. All of these figures show an overall good performance of the proposed method, with level of significance 0.95. Note that the first, second, and third quartiles in these figures are denoted $qL0.25$ and $qU0.25$, $qL0.50$ and $qU0.50$, and $qL0.75$ and $qU0.75$ corresponding to the NPI lower and upper survival functions, respectively. We note that for corresponding proportion with larger values of n, the differences between lower and upper survival functions tend to decrease. That means that when based on more data, the NPI lower and upper survival functions allow more precise inference. From these simulations, we conclude that the proposed approach using the Arrhenius-Weibull model provides suitable predictive inference if the model assumptions are fully correct. We will investigate robustness in case of model misspecification in simulation in the future.

5. Concluding remarks

This paper has presented statistical methods for ALT using the Arrhenius-Weibull model under constant stress testing, supported by the theory of imprecise probability, where the imprecision results from the use of the likelihood ratio test. The proposed method applies the use of the likelihood ratio test to compare the survival distribution of pairwise stress levels, in combination with the Arrhenius model finding the accepted interval of γ values according to the null hypothesis. The observations at the increased stress levels were transformed to interval-valued observations at the normal stress level by developing imprecision in the link function of

(a) Using $\underline{\gamma}$, $\overline{\gamma}$, $n = 10$ (b) Using $\underline{\gamma}$, $\overline{\gamma}$, $n = 50$ (c) Using $\underline{\gamma}$, $\overline{\gamma}$, $n = 100$

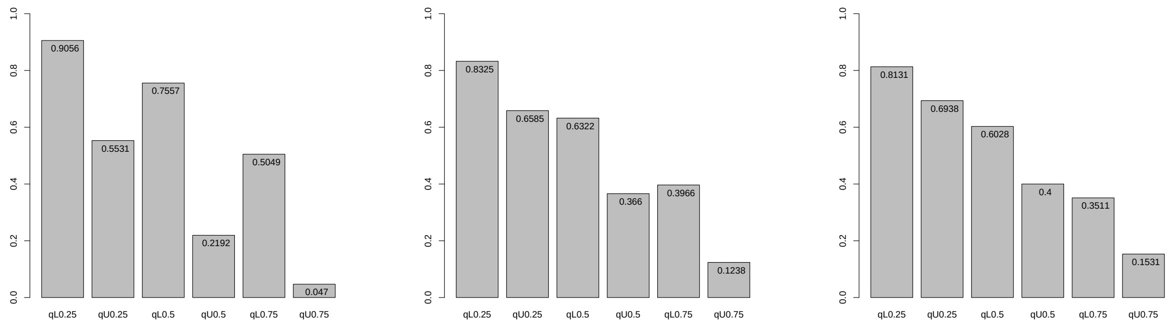

Figure3. Proportion of runs with future observation greater than the quartiles.

the Arrhenius model via a classical test between the pairwise stress levels. We found that using an interval of values for the parameter in the link function between different stress levels enabled us to achieve a greater level of robustness than if we were to use a single point for the parameter. Using the Arrhenius model, we linked the data at different stress levels to the normal stress level, after which NPI can be used at normal stress level.

The use of pairwise tests instead of one test on all stress levels simultaneously, is discussed in detail by [Ahmadini and Coolen 2018], who do not assume a Weibull distribution at each stress level and use the non-parametric log-rank test instead of the likelihood ratio test. The argument for use of the pairwise test is the same: if the model fits poorly, a single test on all stress levels would result in less imprecision while our proposed method, combining pairwise tests tends to, result in more imprecision. The application of our novel method in this paper assumed that we have failure times observed at the normal stress level K_0. The assumption of having failure data at the normal stress level K_0 may not be realistic in real world applications. In this case, we can apply our method with basic function to a higher stress level that is above the normal stress level K_0. The combined data at that level are then transformed all together to the normal stress level K_0. Investigating this is a topic for future research. Also the presented method can be applied similarly if the data contain some right-censored observations, this will also be considered in detail in future research.

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