New method for availability computing of complex systems using Imprecise Markov models

J. Akrouche*, M. Sallak*, E. Châtelet**, F. Abdallah*** and H. Hajj Chehade***

**Sorbonne universités, Université de technologie de Compiègne, CNRS,Heudiasyc,CS 60 319, 60 203 Compiègne cedex, France. (joanna.akrouche, mohamed.sallak)@hds.utc.fr*

*** Université de Champagne, Université de technologie de Troyes, CNRS, Institut Charles Delaunay, 12 rue Marie Curie, CS 42060, 10004 Troyes cedex, France. eric.chatelet@utt.fr*

**** Lebanese University, Beirut-Hadath, Lebanon. (fahed.abdallah76, hiba.hajchhade)@gmail.com*

Abstract. The system availability is defined as the probability that the system is operational at a given time. To compute the availability of a multi-states system (the system and its components could have multi-states), we have to sum over all the probabilities of the components working states, therefore these probability precise values are required. In some cases (rare event failures, new components, ...), it isn't possible to obtain the working probabilities precisely because of the lack of data. In this work, we propose to apply imprecise continuous Markov chain where the failure and repair rates are imprecise. Only few works were developed using this concept. The precise initial distributions are replaced by intervals, which represents the unknown initial probabilities and the unknown transition matrix. The interval constraint propagation method is exploited for the first time, in availability modeling, to compute the imprecise multi-states system availability. The probability interval bounds associated to real variables are contracted, without removing any value that may be consistent with the set of constraints. The proposed methodology is guaranteed, and different examples of complex systems with some properties (convergence, ergodicity, ...) are studied. All the numerical examples and results will be discussed in the paper.

1. Introduction

When we study the dependability of a system, we are confronted in many cases with the fact that the system and its components may have different states or modes of functioning with different performance levels. Such systems are called multi-states systems (MMS). Applied to MSSs, availability analysis calculates the ability of the system to provide a required level of performance based on its level of degradation at a time t. Several methods have been proposed to calculate the availability of an MSS: Universal Generator Function (UGF) [1], inclusion-exclusion method [2], Monte Carlo simulations, and Markovian models[3]. The UGF is applicable only in the case of systems with simple structures: series, parallel, series-parallel, ... The inclusion-exclusion method is adopted only for systems where the components have binary states. Monte Carlo simulations are expensive in terms of time. In this paper, we have chosen to use Markovian model since it is a commonly used method in dependability domain for reliability and availability calculation of an MSS with repair rates [4].

On the other hand, in dependability, the knowledge that we have about the components data is generally uncertain. The validity of the results of the studies depends on the total or partial taking into account of the imperfection of the used knowledge. This requires methods that allow modeling and manipulation of these uncertainties such as probability theory, fuzzy set theory, belief functions theory, … Among the works that have done to calculate the availability of an MSS, those based on the imprecise UGF method [5] using the arithmetic interval probability approach applied on the UGF, and the work proposed by Destercke et al. [6] proposing the use of the belief functions applied on the UGF method. In this work, we propose the use of imprecise Markov models for the calculation of the availability of an MMS. In a previous paper [7], we have some preliminary work about this subject.

2. Problem statement

The main issue in our work is to calculate the availability of complex MSS in presence of different types of uncertainties. Indeed, we are interested in uncertainties about reliability data (failure rate λ and repair rate μ of each component), due to the difficulty to estimate these data (new components, rare components failures, expensive components, …) and the transitions rates could be variable over time and affected by several factors. The proposed way to cope with these uncertainties is to use the theory of imprecise probabilities, and particularly interval probabilities. We consider our transition rates as not being precise, but instead being bounded by intervals. That's why we propose to treat the data that we have in the terms of intervals and to apply interval analysis on imprecise Markov approaches, so we could calculate the imprecise availability of a system.

3. Contribution

In our study, we consider a complex MSS where the components and the system are multi-states and the number of components is important. The repair rates and the failure rates are represented in terms of intervals. To calculate the imprecise availability, we chose to use Markovian approaches.

3.1 *Imprecise Markovian approaches*

Markovian models represent a class of stochastic processes where system is "memoryless" [8], i.e., the future state of the system is independent of the previous states. The time in our case is continuous. The transition rates q_{ij} are in terms of failure rates and repair rates. Q the transition matrix is the matrix containing these elements. Note that the sum of the elements of each line of Q is equal to 0. Since we focus on the imprecision of failure and repair rates, we will take each element q_{ij} as an interval $q_{ij} = [\underline{q_{ij}}\ ;\ \overline{q_{ij}}]$, where $\underline{q_{ij}}$ and $\overline{q_{ij}}$ represent respectively the lower and upper bounds of the interval the interval. Thus, the transition matrix Q will be in the form of an interval $Q = [\underline{Q}\ ;\ \overline{Q}]$.

3.2 *Evaluation of the Availability*

Based on the Chapman-Kolmogorov equations, we get the following formula

$$\dot{P}(t) = P(t).Q \qquad (1)$$

Where $P(t)$ is the probability states vector of the system at a time t. We are interested in the stationary states probabilities that's why we need to compute the asymptotic availability. Using eq.(1), we obtain the following asymptotic solution

$$\Pi.Q = 0 \qquad (2)$$

where $\Pi = [\pi_i]$ is the steady probability vector of the system to be in a state i; $lim_{t\to\infty} P_i(t) = \pi_i$ and we always have the property $\sum_{i=1}^{n} \pi_i = 1$. Eq. (2) is a system of equations that we need to solve in order to find the intervals $\pi_i = [\underline{\pi_i}\ ;\ \overline{\pi_i}]$. Then, the asymptotic availability will be computed as the sum of the probabilities π_i corresponding to the working states. Solving the system of these equations isn't simple, therefore we introduce a method inspired from interval analysis : the technique of contraction [9].

3.2 *The technique of contraction*

The contraction technique [9], was introduced for the first time in 1970 in the artificial intelligence field. It is a method of constraint satisfaction problem and was later developed in interval analysis methods. This technique makes it possible to contract an interval $[x]$ by operators named contractors in order to obtain an interval $[x']$, such that $S \subset [x'] \subset [x]$ where S is the exact interval (note that this technique is more detailed in the appendix). We will use the contractor Forward-Backward propagation [9] for several reasons. It is applicable for all kinds of equations, besides calculations are much simpler than with other contractors, the obtained intervals after the contraction are guaranteed that they belong to the initials intervals. The Forward-Backward propagation technique consists of two steps, the first is the "Forward" : it proposes to make an arithmetic calculation of the intervals. The second step is the "Backward" : it proposes to perform the same arithmetic calculation already made but in the opposite order, which means from the last operation made to the first one, taking into account in each computation step to take the intersections of $[x']$ and $[x]$. In our case the intervals $[x]$ are the $[\pi_i]$ that we need to contract them to obtain the smallest intervals, so we could calculate the availability of the system.

4. Methodology of the proposed method

To calculate the availability of a complex MSS with multi-state components, we first need to illustrate its Markov graph. So, we could present all the transitions from one state to another, from the total working state to the total failure state of the system. We define the interval transition matrix in terms of failure rates and repair rates of dimension ($n\times n$).

To determine the availability of the system, first we take the initial vector Π as Π=[[0,1][0,1] ... [0,1]] , and we apply eq. (2) where we obtain a system of n equations with the last equation representing the fact that $\sum_{i=1}^{n} \pi_i = 1$. We apply the Forward-Backward on the intervals $\pi_i = [\underline{\pi_i}\ ;\ \overline{\pi_i}]$, to reduce Π as much as it is possible. We keep repeating the contraction on all the equations and several times, until the vector Π converges. When we obtain Π, we calculate the imprecise availability of the system A by computing the sum of π_i over all the working states.

To verify if the results obtained by the technique of contraction is accurate, we will compare it to the availability obtained by taking all the possible transition matrix by doing a combination between the lower bounds and upper bounds of the failure rates interval and the repair rates interval, so for each combination we obtain a transition matrix, where we solve the system from eq.(2) and find the corresponding Π and the availability, after that we compare all the obtained availabilities from all combinations, so we can choose the corresponding availability interval where its bounds are the lowest availability (lower bound) and the highest availability (upper bound). We will call this method the “Exact method”. In addition, we can verify the results, by taking the center of each interval of our data, so we will get one transition matrix formed from all the midpoint values of the data, we solve eq(2) to get the precise vector Π, where we could finally calculate the precise availability, which it has to belong to th interval obtained by the contraction technique and to the interval obtained by the exact method. We will call this method the “Precise method”. Figure1 illustrate the steps of the technique of contraction.

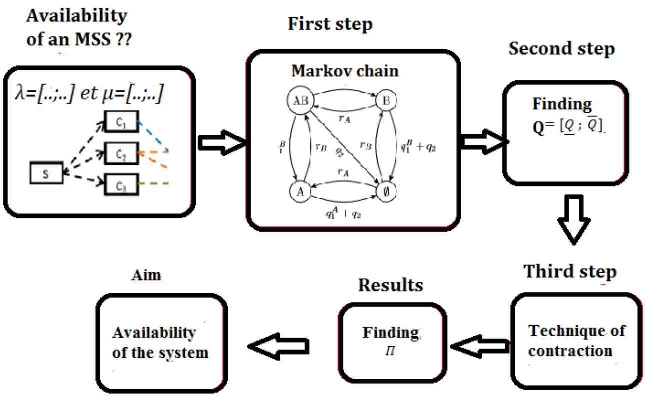

Figure 1. Methodology of the technique of contraction

5. Numerical example

To illustrate the technique of contraction with the Forward-Backward propagation, we will apply it on the following example. We aim to compute the availability of a system presented in Figure 2, and composed of eight components (A, B, C, D, E, F, G and H).

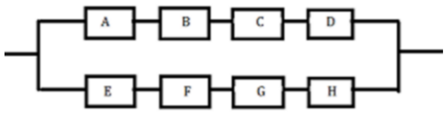

Figure 2. MSS composed of eight components.

The transmission of the system is from left to right. We assume that components B, C, E and H are binaries, i.e., they have only two possible states: a state of total failure, and a state of full functioning. Components A, F and G have three possible states: a state of total failure, a state of full functioning, and a state of partial failure. Component D has four possible degraded states: a state of total failure, a state of full functioning, a state of degraded functioning of type 1, and a state of degraded functioning of type 2. Thus, the total number of states for the system is 1728. All the information about repair rates and failure rates of each components are presented in Table 1.

Component A	$\lambda_{1,1} = [3\times10^{-5}; 4.5\times10^{-5}]$ $\lambda_{1,2} = [2.6\times10^{-5}; 7\times10^{-5}]$	$\mu_{1,1} = [4.1\times10^{-1}; 7\times10^{-1}]$ $\mu_{1,2} = [3\times10^{-1}; 6.2\times10^{-1}]$
Component B	$\lambda_2 = [2.6\times10^{-5}; 3.1\times10^{-5}]$	$\mu_2 = [1.6\times10^{-1}; 4\times10^{-1}]$
Component C	$\lambda_3 = [3.5\times10^{-5}; 4.6\times10^{-5}]$	$\mu_3 = [2.1\times10^{-1}; 3\times10^{-1}]$
Component D	$\lambda_{4,1} = [2.6\times10^{-5}; 3.5\times10^{-5}]$ $\lambda_{4,2} = [2.9\times10^{-5}; 3.7\times10^{-5}]$ $\lambda_{4,3} = [2.7\times10^{-5}; 3.4\times10^{-5}]$	$\mu_{4,1} = [3.1\times10^{-1}; 3.5\times10^{-1}]$ $\mu_{4,2} = [2.9\times10^{-1}; 3.3\times10^{-1}]$ $\mu_{4,3} = [3\times10^{-1}; 3.5\times10^{-1}]$
Component E	$\lambda_5 = [3\times10^{-5}; 4.1\times10^{-5}]$	$\mu_5 = [2.5\times10^{-1}; 3\times10^{-1}]$
Component F	$\lambda_{6,1} = [3.1\times10^{-5}; 5.6\times10^{-5}]$ $\lambda_{6,2} = [2.9\times10^{-5}; 4.6\times10^{-5}]$	$\mu_{6,1} = [2\times10^{-1}; 2.4\times10^{-1}]$ $\mu_{6,2} = [2.6\times10^{-1}; 3\times10^{-1}]$
Component G	$\lambda_{7,1} = [4.1\times10^{-5}; 4.7\times10^{-5}]$ $\lambda_{7,2} = [2.9\times10^{-5}; 3.2\times10^{-5}]$	$\mu_{7,1} = [2.3\times10^{-1}; 2.9\times10^{-1}]$ $\mu_{7,2} = [2.7\times10^{-1}; 3.1\times10^{-1}]$
Component H	$\lambda_8 = [4.1\times10^{-5}; 5\times10^{-5}]$	$\mu_8 = [2.7\times10^{-1}; 3\times10^{-1}]$

Table 1. Failure rates and repair rates form a state to another for each components

Where $\lambda_{k,i}$ and $\mu_{k,i}$ represent the failure rate *i* and repair rate *i* of component *k*.

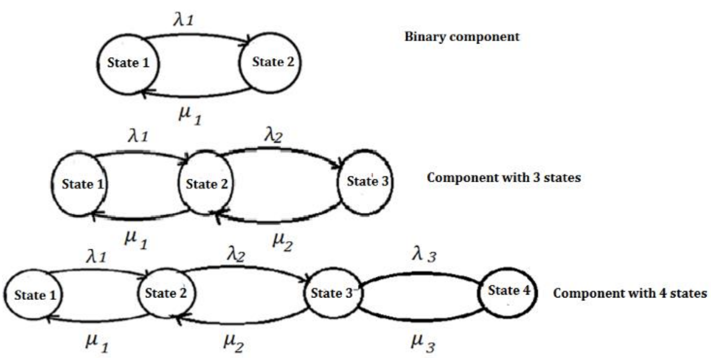

Figure 3. The corresponding Markov graph for each component

Since in this example we consider that the components are independents, so for each component, we will construct its Markov graph (cf. Figure 3), to determine its interval transition matrix. Then, we will apply eq.(2) to find Π_k for each component, with $k = \{A, \dots, H\}$. Finally, we compute the availability of each component by computing the sum of $\Pi_{k,i}$ over all the working states. In this example, the availability of the entire system A_{system} is presented in the following equation:

$$A_{system} = 1 - (1 - A_A A_B A_C A_D)\,(1 - A_E A_F A_G A_H) \quad (3)$$

To find Π_k for each component we will apply the technique of contraction. We propose to compare the obtained availability with the availability of the system obtained by the exact method and by the precise one. All the results are presented in Table 2.

Contraction technique	Exact method	Precise method
[0.9999991; 0.999999999]	[0.9999996;0.999999995]	0.99999992

Table 2. The availability of the system by applying different methods.

6. Conclusion

The main contribution in this paper is to compute the availability of a complex MSS. We have proposed the use of imprecise Markovian approaches in presence of imprecise failure and repair rates. Using the Contraction methods, and particularly the forward-backward contraction method on the equations obtained by the transition matrix of a Markov chain representing the MSSs. The main goal of the Contraction method is to reduce the initial interval to its most possible minimum size. After testing the method on several examples, one of them the one presented in this paper, and comparing the results of the obtained availability by the exact method and the precise one, we found that the interval obtained by the technique of contraction is more conservative than the interval obtained by the exact method. We have also find that precise availability is always belong to the intervals. Based on this method, we can study the availability of larger systems, where the number of components and the number of possible states for each element is higher, and then we can proceed to our future objective: the optimization of the imprecise availability of systems when considering imprecise failure and repair rates.

Appendix

Consider n_x variables $x_i \in \mathbb{R}$, $i = 1,..,n_x$, linked by n_f relations (or constraints) [9] of the form: $\boldsymbol{f}_j(x_1,..,x_{nx}) = 0$, $j \in 1,..,n_f$, it can be written in vector notation as: f(x) = 0. Each variable x_i is known to belong to a domain X_i. For simplicity, these domains will be intervals, denoted by $[x_i]$. Define the vector $\boldsymbol{x}= (x_1,..,_{nx})^T$ and the prior domain for $\boldsymbol{x}$ is a box as: $[\boldsymbol{x}] = [x_1] \times \dots \times [\,x_{nx}]$. Let $\boldsymbol{f}$ be the function whose coordinate functions are the f_j's. Equation f(x) = 0 corresponds to a constraint satisfaction problem (CSP) H, which can be formulated as: (H: f(x) = 0 , $\in [\boldsymbol{x}]$). The solution set of H is defined as: $S = \{\boldsymbol{x} \in [\boldsymbol{x}] | \boldsymbol{f}(\boldsymbol{x}) = 0\}$. Contracting H means replacing $[\boldsymbol{x}]$ by a smaller domain $[\boldsymbol{x}']$ such that the solution set remains unchanged, i.e. $S \subset [\boldsymbol{x}'] \subset [\boldsymbol{x}]$. There exists an optimal contraction of H, which corresponds to replacing $[\boldsymbol{x}]$ by the smallest box that contains S. A contractor for H is any operator that can be used to contract it. A contractor C is defined as an operator used to contract the initial domain of the CSP, and thus to provide a new box. Several contractors exist, each works in a different manner and is efficient only for specific CSPs and for certain cases [9]. The most popular contractor, which will be used in our approach, is the "Forward-backward propagation (FBP) contractor". This technique is known for its simplicity and easiness, it is also more general than the others since it works on all type of systems [9]. It also gives guaranteed results which means that during the contraction we always get an interval belonging to the initial interval. The Forward-backward (FBP) contractor $C\downarrow\uparrow$, is a classical algorithm in constraint programming for contracting. This contractor makes it possible to contract the domains of the CSP H by taking into account each one of the n_f constraints apart. The algorithm works in two steps [9]. The forward step applies interval arithmetic to each operator of the function y=f(x), from the variable's domain ([x]) up to the function's domain ([y]), this step considers the direct forms of the equations. The backward step sets the interval associated to the new function's domain [y] to [0, 0] (imposes constraint satisfaction, since we are solving f(x)=0) and, then, applies backward arithmetic from the function's domain to the variable's domain, which means using the inverse of the functions that appear in the equations f(x). The following example explains the procedure of the FBP technique. *Example:* Consider the constraint $y = -5x_1 + 2x_2 = 0$ and the initial box-domain $[x] = [1, 4] \times [-3, 7]$. This constraint can be decomposed as shown in eq.(4) into three primitive constraints (i.e. constraints associated with a unique elementary function: multiplication or addition) by introducing two intermediate variables a_1 and a_2 defined as: $a_1 = -5x_1$ and $a_2 = 2x_2$. Initial domains for these variables are determined as follows:

$$a_1 = -5x_1 = -5 \times [1,4] = [-20,-5]$$
$$a_2 = 2x_2 = 2 \times [-3,7] = [-6,14]$$
$$y = a_1 + a_2 = [-20,5] + [-6,14] = [-26,9] \tag{4}$$

and this step is called the "forward propagation". A method for contracting H with respect to the constraint $f(x) = 5x_1 + 2x_2 = 0$ is to contract each of the primitive constraints in eq.(4) until the contractors become inefficient. For this example: Since f(x) = 0, the domain for y should be taken equal to {0}, so we can add the step: $[y] := [y] \cap \{0\}$. If [y] as computed in eq.(4) turns out to be empty, then the CSP has no solution. Else, [y] is replaced by 0, which is the case in this example. After, a backward propagation is performed, updating the domains associated with all the variables to get:

$$[a1] := ([y] - [a2]) \cap [a1] = [-14, -5]$$
$$[a2] := ([y] - [a1]) \cap [a2] = [5,14]$$
$$[x1] := ([a1]) / -5) \cap [x1] = [1,14/5]$$
$$[x2] := ([a2]) / 2) \cap [x2] = [5/2,7] \tag{5}$$

Thus, we obtain the new box: $[x](1) = [1, 14/15] \times [5/2, 7]$, which is the result of the first FBP contraction. Iterating this procedure, the resulting sequence of boxes [x](k) converges towards the smallest possible domain, after which the domains no longer change following another iteration of FBP.

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