Modeling spare part demand data using transmuted or exponentiated Poisson distribution

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Abstract. Traditionally, it is assumed that the lead time spare part demand follows the Poisson or normal distribution. It is well-known that the mean of Poisson distribution equals to its variance, which is called Poisson dispersion. Actual demand data scarcely meet the Poisson dispersion. To overcome this problem, several two-parameter variants (e.g., zero-inflated Poisson and hurdle shifted Poisson) of the Poisson distribution have been used for modelling irregular spare part demand data. In this paper, we propose two new two-parameter alternatives, which are transmuted and exponentiated Poisson distributions. The proposed distributions can be over- or under-Poisson dispersive, and hence provide better flexibility without losing the simplicity of the Poisson distribution too much. The appropriateness of the proposed models is illustrated through three real-world examples. Based on the idea that a fitted distribution should provide a good goodness-of-fit to the right tail of the empirical demand distribution so as to achieve good inventory performance, a weighted least square method is also proposed to estimate the model parameters. The appropriateness of the proposed approach is also illustrated.
Keywords. Spare parts demand, transmuted Poisson distribution, exponentiated Poisson distribution, weighted least square method

1. Introduction

The determination of the optimal stock level in an inventory management problem needs to appropriately use a demand distribution (Costantino et al. 2017). The standard assumption is that lead time demand follows a normal or Poisson distribution. This is often not true, particularly for high irregular demand patterns (e.g., see Turrini and Meissner 2017). The departures from the assumed distribution may lead to very unsatisfactory inventory performance, e.g., high inventory costs (Nenes et al. 2010, Turrini and Meissner 2017).

A number of different distribution models have been used in the literature, including the stuttering Poisson (or geometric Poisson e.g., see Chen 2012), zero-inflated Poisson (ZIP), hurdle shifted Poisson (HSP), negative binomial, normal and gamma distributions (e.g., see Snyder et al. 2012, Costantino et al. 2017, Turrini and Meissner 2017). Among these distributions, the stuttering Poisson is computationally intensive, ZIP, HSP and negative binomial distribution are only applicable for over-Poisson dispersive cases and the normal and gamma distributions are continuous distributions.

In this paper, we present two two-parameter extended Poisson distributions: exponentiated Poisson and transmuted Poisson distributions as alternative distributions for modelling spare parts demand. The exponentiated Poisson was previously used in modelling bus-motor failure data (Jiang 2010). In this paper, we apply it to model spare parts demand data. We will show that it provides very good goodness-of-fit to the spare parts demand data. The transmuted Poisson distribution is a new model. We study its properties and examine its appropriateness in modelling spare parts demand data.

According to Turrini and Meissner (2017), most inventory applications involve calculating an upper percentile (e.g., 98%) of the forecasted demand distribution, implying that the right tail of the distribution provides more information for inventory management. Based on this observation, we propose a weighted least square method (LSM) for fitting a distribution to a set of demand data in order to achieve good inventory performance. We use a normal distribution function as the weight function, whose parameters depend on the sample mean. The fitted model obtained from the weighted method is expected to improve the inventory performance.

The paper is organized as follows. In Section 2, we summarize two-parameter extensions of the Poisson distribution and study the properties of the transmuted Poisson distribution. The weighted LSM is presented in Section 3. The appropriateness and usefulness of the proposed models and approach are illustrated in Section 4. The paper is concluded in Section 5.

2. Proposed model and its properties

2.1 *Earlier extensions of the Poisson distribution*

There are several extended Poisson distributions and those extensions with two parameters are outlined as follows.

The first extension is the stuttering Poisson distribution, which is obtained by compounding the Poisson distribution with a geometric distribution (Chen 2012). Its probability mass function (pmf) is given by

$$f(x)=\begin{cases} e^{-\lambda},\ for\ x=0 \\ \sum_{i=1}^{x} C_{x-1}^{i-1}(1-\rho)^i \rho^{x-i}\dfrac{\lambda^i e^{-\lambda}}{i!},\ for\ x>0 \end{cases},\ \lambda>0,\ \rho\in(0,1). \tag{1}$$

The mean and variance are given by

$$\mu=\lambda/(1-\rho),\ \sigma^2=\mu(1+\rho)/(1-\rho). \tag{2}$$

From equation (2), we have $\sigma^2>\mu$, implying that it is always over-Poisson dispersive. Clearly, it is computationally intensive and hence its application is limited.

The second extension is the zero-inflated Poisson distribution. It is developed for modeling the excess zeros (e.g., see Lambert 1992, Majeske and Herrin 1995, and Costantino et al. 2017). The pmf of the ZIP model is given by

$$f(x)=\begin{cases} 1-\alpha+\alpha e^{-\lambda},\ for\ x=0 \\ \alpha\dfrac{\lambda^x e^{-\lambda}}{x!},\ for\ x>0 \end{cases},\ \lambda>0,\ \rho\in(0,1). \tag{3}$$

The mean and variance are given by

$$\mu=\alpha\lambda,\quad \sigma^2=\mu(1+\lambda-\alpha\lambda). \tag{4}$$

Clearly, we have $\sigma^2>\mu$, implying that it is always over-Poisson dispersive.

The third extension is the hurdle shifted Poisson distribution (SPD), which is a mixture of a deterministic (or degenerate) distribution and the shifted Poisson distribution (Snyder et al. 2012). Its pms is given by

$$f(x)=\begin{cases} q,\ for\ x=0 \\ (1-q)\dfrac{\lambda^{x-1}e^{-\lambda}}{(x-1)!},\ for\ x>0 \end{cases}. \tag{5}$$

The mean and variance are given by

$$\mu=(1-q)(\lambda+1),\quad \sigma^2/\mu=q(1+\lambda)+\lambda/(1+\lambda). \tag{6}$$

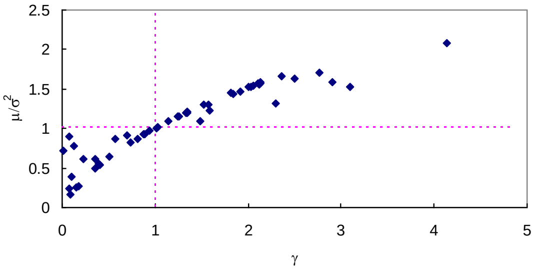
Clearly, it can be over-Poisson dispersion (when λ or q is large) or under-Poisson dispersion (when λ and q are small).

2.2 *Proposed extensions of the Poisson distribution*

The first author proposes an exponentiated Poisson distribution (EPD) for modelling bus-motor failure data (Jiang 2010). The EPD is defined by

$$F(x)=e^{-\lambda\gamma}\left(\sum_{i=0}^{x}\frac{\lambda^i}{i!}\right)^{\gamma},\ \lambda,\gamma>0. \tag{7}$$

To examine its Poisson dispersion, we randomly generate a set of combinations of λ and γ. For each combination, we calculate the values of μ and σ^2. Figure 1 shows the plot of μ/σ^2 vs. γ, and Figure 2 shows the plot of μ vs. λ. As can be seen from these two figures, the EPD can be either over-Poisson dispersion (when γ < 1) or under-Poisson dispersion (when γ > 1); and the mean demand (μ) is highly correlated with the value of λ.



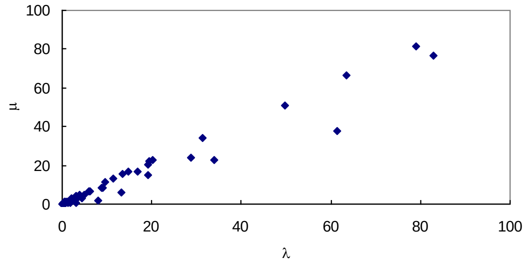


Figure 1 Plot of μ/σ^2 vs. γ Figure 2 Plot of μ vs. λ

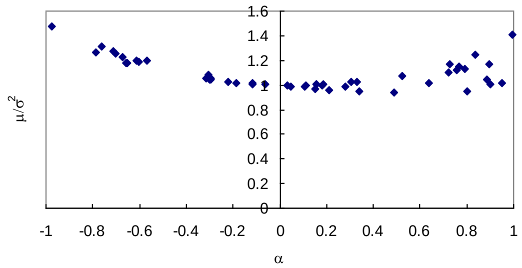
2.3 Transmuted Poisson distribution

For an arbitrary distribution function $G(x)$, Shaw and Buckley (2009) define a transmuted distribution as

$$F(x) = (1+\alpha)G(x) - \alpha G^2(x), \quad |\alpha| \le 1. \tag{8}$$

When $G(x)$ is the Poisson distribution, we obtain the transmuted Poisson distribution (TPD). In this paper we examine the appropriateness of TPD as an alternative distribution to model the demand data.

To examine its Poisson dispersion, we randomly generate a set of combinations of λ and α. For each combination, we calculate the values of μ and σ^2. Figure 3 shows the plot of μ/σ^2 vs. α; and Figure 4 shows the plot of μ vs. λ. As can be seen from these two figures, TPD can be slightly over-Poisson dispersive (when $\alpha > 0$ and λ is relatively small) or under-Poisson dispersion; and the mean demand (μ) is highly correlated with the value of λ.



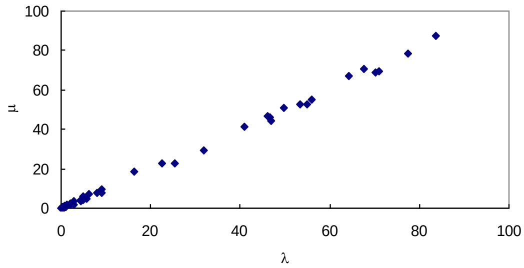


Figure 3 Plot of μ/σ^2 vs. α Figure 4 Plot of μ vs. λ

3. Weighted least square method

Suppose that we have a set of demand data given by

$$(n_i, 0 \le i \le m). \tag{9}$$

The empirical distribution function is given by

$$F_i = \sum_{j=0}^{i} n_j / \sum_{j=0}^{m} n_j, 0 \le i \le m; \quad F_i = 1, i > m. \tag{10}$$

Consider a parametric model $F(i;\theta)$, where θ is the parameter set to be estimated. To stress the observations with large value of i, we introduce a weight function $w(i)$, which is positive and increases with i. In this paper, we define the weight function as

$$w(i) = \Phi(i; \mu_s, \sqrt{\mu_s}) \tag{11}$$

where $\Phi(i; \mu_s, \sqrt{\mu_s})$ is the normal distribution with mean and variance being the sample average μ_s. The parameters are estimated through minimizing the sum of weighted square errors (SWSE)

$$SWSE = \sum_{i=0}^{N} w(i)[F(i;\theta) - F_i]^2, N > m. \tag{12}$$

We use $1 - F_{i_{0.98}}$ to evaluate the performance of the fitted model, where $i_{0.98}$ is the 98 percentile of the fitted distribution. A good fit implies:

$$1 - F_{i_{0.98}} \approx 0.02. \tag{13}$$

4. Illustrations

In this section, we present two real-world examples to illustrate the appropriateness of the proposed models and weighted approach.

4.1 Example 1

The data shown in Table 1 come from Kogan and Rind (2011) and deal with seven years historical consumption of 25 kV distribution transformers.

Table 1 Demand data for Example 1

	2001	2002	2003	2004	2005	2006	2007
1	168	30	8	5	7	10	4
2	264	12	12	0	29	8	2
3	106	0	12	13	6	8	0
4	65	0	20	9	18	0	1
5	213	62	23	7	28	0	19
6	179	49	13	20	20	13	16
7	156	30	29	14	21	4	1
8	162	65	17	19	21	2	20
9	67	22	7	3	0	0	27
10	102	5	10	6	0	1	3
11	7	24	13	0	11	0	28
12	23	14	4	2	15	0	1

It is noted that the spare parts consumed in 2001 is much more than the spare parts consumed in 2002. Kogan and Rind (2011) think that the cause is due to the impact of extreme weather events. In fact, the cause may also be due to earlier failures. In this paper, we exclude the data in 2001.

The lead-time is 8 months. We consider all the sums in any successive eight months. Table 2 shows the lead time demand data. The sample mean and variance are 98.64 and 3724.55, respectively. Clearly, the data are highly over-Poisson dispersive.

Table 2 Lead time demand data

248	271	117	134	116	61	87	71	71	150	116	52	45	20	8	63	115
240	217	118	133	98	61	85	71	96	143	98	52	35	24	27	86	
233	180	126	131	85	71	91	80	110	114	86	65	28	13	42	87	
257	162	131	132	69	72	78	72	131	119	73	58	20	9	43	115	

Table 3 shows the parameters estimated from the weighted least square method (WLS) and corresponding performances in terms of $1-F_{i_{0.98}}$. To illustrate the advantage of the proposed parameter estimation method, we also show the parameter and corresponding performance of the Poisson distribution (in the second row), obtained from the moment method (MM). From the table, we have the following observations:

- According to the results in the second and third rows, the model fitted from the WLS method outperforms the model fitted from the moment method in terms of $1-F_{i_{0.98}}$.
- TPD offers the best goodness-of-fit in terms of SWSE and $1-F_{i_{0.98}}$.
- The performance of TPD is almost the same as the performance of the Poisson distribution. This is because the data is highly over-Poisson dispersive.

Table 3 Estimated parameters and model performances for Example 1

Model	Method	λ	α, γ, q	SWSE	$i_{0.98}$	$1-F_{i_{0.98}}$
Poisson	MM	98.6		3.192	120	0.2615
Poisson	WLS	104.7		2.939	126	0.2462
ZIP	WLS	139.3	0.3539	0.9269	158	0.1231
SPD	WLS	138.3	0.6462	0.9284	158	0.1231
EPD	WLS	172.4	0.0301	0.7092	173	0.1077
TPD	WLS	105.3	0.1067	2.934	126	0.2462

4.2 Example 2

The data shown in Table 4 come from Guo et al. (2017) and deal with the annual total spare parts turnover numbers (which reflect the actual demand of a repairable unit) of Parts 1 and 2. The sample mean and variance for Part 1 [Part 2] are 5.4 and 3.14 [9.3 and 12.70], respectively, implying under-Poisson [over-Poisson] dispersion.

Table 5 shows the parameters estimated from the MM (for the Poisson distribution) and WLS method as well as the corresponding performances. From the table, we have the following observations:

- According to the results in the 3rd and 4th rows as well as the 11th and 12th rows, the model fitted from WLS method outperforms the model fitted from MM in terms of SWSE, $i_{0.98}$ and $1-F_{i_{0.98}}$.

- For the demand data of Parts 1 and 2, EPD offers the best goodness-of-fit.
- For the demand data of Part 1, TPD is the second best model. This is because this data set is under-Poisson dispersive.
- For the demand data of Part 2, TPD is the third best model (slightly poorer than the ZIP). This is because this data set is over-Poisson dispersive.

This illustrates the appropriateness of the proposed models and estimation approach.

Table 4 Demand data for Example 2

Year	Part 1	Part 2	Year	Part 1	Part 2	Year	Part 1	Part 2	Year	Part 1	Part 2
1	7	11	14	6	10	27	6	11	40	3	11
2	6	9	15	8	9	28	5	6	41	4	8
3	8	10	16	9	16	29	4	4	42	4	7
4	6	8	17	5	18	30	4	5	43	3	6
5	5	8	18	8	10	31	5	5	44	4	5
6	10	9	19	7	18	32	3	12	45	4	7
7	7	12	20	4	12	33	3	7	46	3	5
8	6	7	21	6	10	34	5	9	47	5	13
9	5	5	22	5	8	35	7	8	48	3	8
10	5	6	23	7	9	36	8	15	49	4	9
11	6	6	24	5	7	37	4	17	50	4	7
12	4	13	25	4	7	38	7	9			
13	4	8	26	9	8	39	6	17			

Table 5 Estimated parameters and model performances for Example 2

Model	Method	λ	α, γ, q	SWSE, 10^{-3}	$i_{0.98}$	$1-F_{i_{0.98}}$
	Part 1					
Poisson	MM	5.400		7.074	11	0
Poisson	WLS	5.193		4.525	10	0
ZIP	WLS	n/a (□□> 1)				
SPD	WLS	n/a (q < 0)				
EPD	WLS	4.330	1.818	0.9993	10	0
TPD	WLS	6.704	0.9866	1.428	10	0
	Part 2					
Poisson	MM	9.300		23.93	16	0.08
Poisson	WLS	9.112		22.58	16	0.08
ZIP	WLS	9.557	0.8844	20.19	16	0.08
SPD	WLS	8.712	0.1443	22.56	16	0.08
EPD	WLS	11.42	0.4368	12.67	17	0.04
TPD	WLS	9.660	0.3004	20.90	16	0.08

5. Conclusions

In this paper we have proposed to use the transmuted and exponentiated Poisson distributions as alternative models for modelling spare parts demand data. We have fitted these two models, along with the Poisson distribution as well as zero-inflated and hurdle shifted Poisson distributions, to three sets of demand data and found that the exponentiated Poisson distribution offers the best goodness-of-fit and the transmuted Poisson distribution is the second best model for under-Poisson dispersive demand data. The main advantages of the exponentiated and transmuted Poisson distributions are their flexibility and relative simplicity.

To improve the inventory performance, we have proposed a weighting approach to estimate the parameters of a distribution model. Its appropriateness has been confirmed.

Topics for future study include

- To carry out more case studies to further check their appropriateness for modelling spare part demand data,
- To examine the inventory performance when the proposed models are used to model the lead time demand, as conducted by Turrini and Meissner (2018),
- To compare the proposed models with other models such as the negative binomial distribution,
- To optimize the parameters of the weight function, and
- To apply the proposed models to other areas.

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References

Chen CW (2012) Using geometric Poisson exponentially weighted moving average control schemes in a compound Poisson production environment. *Computers and Industrial Engineering*, 63 (2), pp. 374-381.

Costantino F, Gravio GD, Patriarca R, Petrella L (2017) Spare parts management for irregular demand items. *Omega*, in press, Available online 28 September 2017.

Guo F, Diao J, Zhao QH, Wang DX, Sun Q (2017) A double-level combination approach for demand forecasting of repairable airplane spare parts based on turnover data. *Computers and Industrial Engineering* 110, pp. 92-108.

Jiang R (2010) Discrete competing risk model with application to modeling bus-motor failure data, *Reliability Engineering and System Safety*, pp. 981-988.

Kogan V, Rind T (2011) Determining critical power equipment inventory using extreme value approach and an auxiliary Poisson model. *Computers and Industrial Engineering*, 60(1), pp. 25-33.

Lambert D (1992) Zero-inflated Poisson regression, with an application to defects in manufacturing. *Technometrics,* 34, pp. 1–14.

Majeske KD, Herrin GD (1995) Assessing mixture-model goodness-of-fit with an application to automobile warranty data. In: *Proceedings of Annual Reliability and Maintainability Symposium*, pp. 378-383.

Nenes G, Panagiotidou S, Tagaras G (2010) Inventory management of multiple items with irregular demand: A case study. *European Journal of Operational Research*, 205(2), pp. 313-324.

Shaw WT, Buckley IR (2009) The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic normal distribution from a rank transmutation map.

Snyder RD, Ord JK, Beaumont A (2012) Forecasting the intermittent demand for slow-moving inventories: A modelling approach. *International Journal of Forecasting* 28, pp. 485-496.

Turrini L, Joern Meissner JM (2017) Spare parts inventory management: new evidence from distribution fitting. *European Journal of Operational Research,* in press, Available online 29 September 2017.