Cost effective component swapping to increase system reliability

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Abstract

One of the strategies that might be considered to enhance reliability and resilience of a system is swapping components when a component fails, so replacing it by another component from the system which is still functioning. This paper considers cost effective component swapping to increase system reliability. The cost is discussed in two scenarios, namely fixed cost and time dependent cost for system failure.

1. Introduction

As an alternative to the use of additional components to provide increased redundancy, the use of standby components, maintenance activities or increased component reliability, Najem and Coolen (2018) introduce component swapping, this is the possibility to replace a failed component by another component in the system which has not yet failed. This is logically restricted to components which are of the same type. It seems that increasing in system reliability through such component swapping has not received much attention in the literature, yet in some scenarios it can be an attractive opportunity to prevent a system from failing. In practice, if this activity can be done at low cost, this could enable preparation of substantial repair activities, or it may be deemed to leave the system reliable enough to complete its mission.

Samaniego (1985) introduced the system signature as a tool for a assessment for systems consisting of components of a single type, by modelling the structure of a system and separating it from the random failure time of components. Coolen and Coolen-Maturi (2012) introduced the survival signature as an alternative to the signature, which can fulfil a similar role to the signature, but can also be used for reliability assessment for systems with multiple types of components. Consider a system that consists of m components of $K \geq 2$ types, with m_k components of type $k \in \{1,2,...,K\}$ and $\sum_{k=1}^{K} m_k = m$. Assume that the random failure times of components of the same type are exchangeable, while full independence is assumed for the random failure times of components of different types. Let the state vector $\underline{x}^k = \left(x_1^k, x_2^k, ..., x_{m_k}^k\right) \in \{0,1\}^{m_k}$ be the state vector representing the state of the system components of type k, with $x_i^k = 1$ if the ith component of type k functions and $x_i^k = 0$ if not. The labelling of the components is arbitrary but must be fixed to define $\underline{x}^k$. Let $\underline{x} = \left(\underline{x}^1, \underline{x}^2, ..., \underline{x}^K\right) \in \{0,1\}^m$ be the state vector for the overall system. The structure function $\phi : \{0,1\}^m \to \{0,1\}$, defined for all possible $\underline{x}$, takes the value 1 for a particular state vector $\underline{x}$ if the system functions and 0 if the system does not function for the state vector $\underline{x}$. The survival signature is defined as the probability that the system functions, given that exactly l_k of type k components function, for $l_k = 0,1,...,m_k$, for each $k = 1,2,...,K$, and is denoted by $\Phi(l_1, l_2, ...l_K)$ (Coolen and Coolen-Maturi, 2012).

There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with exactly l_k of its m_k components $x_i^k = 1$, so with $\sum_{i=1}^{m_k} x_i^k = l_k$. We denote the set of these state vectors for components of type k by S_l^k. Let $S_{l_1,...,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, for $k = 1,2,...,K$. Because of the assumption that the failure times of m_k components of type k are exchangeable, all the state vectors $\underline{x}^k \in S_l^k$ are equally likely to occur, Thus, $\Phi(l_1, l_2, ...l_K)$ is given by following equation:

$$\Phi(l_1, l_2, ...l_K) = \left(\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right) \times \sum_{\underline{x} \in S_{l_1,...,l_K}} \phi(\underline{x}) \tag{1}$$

Let $C_t^k \in \{0,1,...,m_k\}$ denote the number of type k components in the system that function at time $t > 0$. The reliability of the system $R_s(t) = P(T_S > t)$ is

$$R_s(t) = \sum_{l_1=0}^{m_1} ... \sum_{l_K=0}^{m_K} \left[\Phi(l_1, ...l_K) \prod_{k=1}^{K} P(C_t^k = l_k)\right] \tag{2}$$

If we further assume, in addition to exchangeability of failure times of components of the same type, that these failure times are conditionally independent and identically distributed, with the probability distribution for the failure time of components of type k specified by the cumulative distribution function (CDF) $F_k(t)$, then

$$R_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_K=0}^{m_K} \left[\Phi(l_1, \dots l_K) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right] \tag{3}$$

Coolen and Najem (2018) implement this concept of survival signature to quantify the reliability of systems if some components can be swapped. In this paper we aim to include considerations of costs. In section 2 of this paper, we first provide a brief introductory overview of the concept of components swapping. In section 3, we consider the cost of component swapping with fixed penalty for system failure and in section 4, we consider the cost of component swapping with time dependent penalty for system failure. In each section, we discuss some related examples. We end the paper with some concluding remarks.

2. Components swapping

Assume that there are fixed swapping rules, which prescribe upon failure of a component precisely which other component takes over its role in the system, if possible and if the other component is still functioning. Assume further that such a swap of components takes neglectable time and does not affect the functioning of the component that changes its role in the system nor its remaining time till failure. Under these assumptions, the effect of such a component swap can be reflected through the system structure function, and hence it can be taken into account for computation of the system reliability through the survival signature.

If we define a swapping case i as a regime of specified swaps that will occur if specific components fail, then let $\phi_{C_i}(\underline{x})$ denote the system structure function. Compared to the system's structure function without swapping opportunities, $\phi(\underline{x})$, ϕ_{C_i} will typically be equal to 1 for some $\underline{x}$ for which ϕ was equal to 0, reflecting the benefit from component swapping upon failure of a component. Let $\Phi_{C_i}(l_1, l_2, \dots l_K)$ denote the survival signature given the defined swapping case i in place, so

$$\Phi_{C_i}(l_1, l_2, \dots l_K) = \left(\prod_{a=1}^{K} \binom{m_k}{l_k}^{-1} \right) \times \sum_{\underline{x} \in S_{l_1, \dots, l_K}} \phi_{C_i}(\underline{x}). \tag{4}$$

Let T_S^i denote the random system failure time with swapping case i in place. Then, the reliability of the system with swapping case i in place $R_{s,i}(t) = P(T_S^i > t)$ is

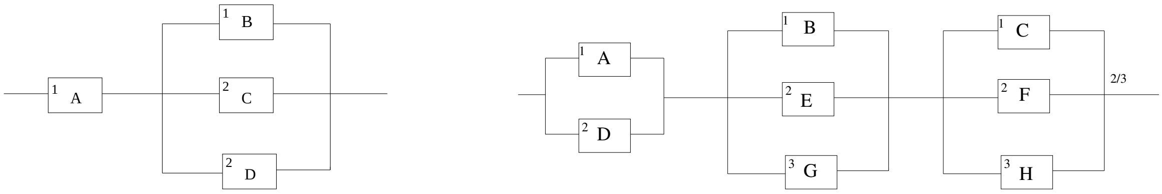
$$R_{s,i}(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_K=0}^{m_K} \left[\Phi_{C_i}(l_1, \dots l_K) \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right] \tag{5}$$

It is important to notice here that the swapping case i is entirely reflected in the system survival signature. Crucially, the components have kept the same failure time distributions and the same assumptions apply, that is failure times of components of the same type remain conditionally independent and identically distributed, and failure times of components of different types remain independent. Hence, we can obtain the difference in reliability by the following equation:

$$R_{s,i}(t) - R_s(t) = \sum_{l_1=0}^{m_1} \dots \sum_{l_K=0}^{m_K} \left[\left[\Phi_{C_i}(l_1, \dots l_K) - \Phi(l_1, \dots l_K) \right] \prod_{k=1}^{K} \left(\binom{m_k}{l_k} [F_k(t)]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right] \tag{6}$$

2.1. *Example*

Consider the system in Figure 1(a), which consists of four components of two types, $m_1 = m_2 = 2$. We want to examine the reliability of this system in the case that if the component A fails but the components B still functions, component A will be swapped by B, we refer to this case as case 1. We refer to the original case of the system when there is no swapping as case 0. $\Phi(l_1, l_2)$ and $\Phi_{C_1}(l_1, l_2)$, are given in Table 1, presenting only non zero values.



(a) System for example 2.1. (b) System for example 2.2.

Figure1. Systems for examples.

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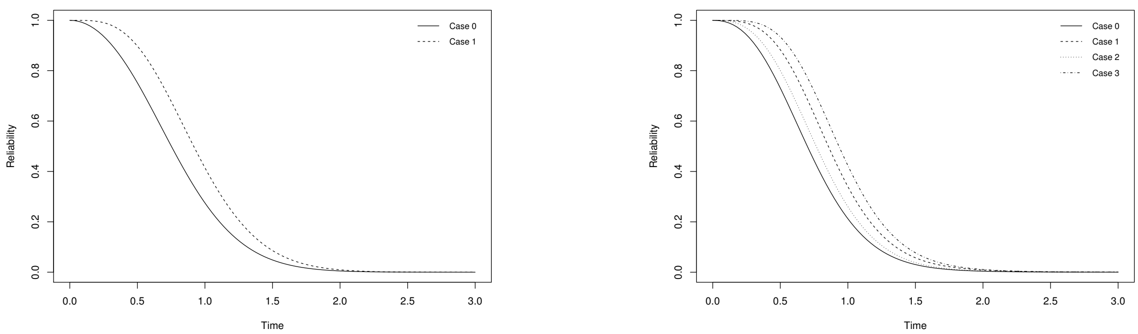
l_1	l_2	Φ	Φ_{C_1}	l_1	l_2	Φ	Φ_{C_1}	l_1	l_2	Φ	Φ_{C_1}
1	1	1/2	1	2	0	1	1	2	2	1	1
1	2	1/2	1	2	1	1	1				

Table 1. $\Phi(l_1, l_2)$ and $\Phi_{C_1}(l_1, l_2)$ of system in Figure 1(a)



(a) For the system in Figure 1(a). (b) For the system in Figure 1(b).

Figure2. The reliability before and after swapping cases.

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Let the probability distributions of the failure times of the type 1 and 2 components have CDF $F_1(t)$ and $F_2(t)$, respectively. The reliability of the system in case 0 is $R_s(t) = [F_1(t)][1-F_1(t)][1-[F_2(t)]^2] + [1-F_1(t)]^2$ and in case 1 is $R_{s,1}(t) = 2[F_1(t)][1-F_1(t)][1-[F_2(t)]^2] + [1-F_1(t)]^2$. Figure 2(a) presents $R_s(t)$ and $R_{s,1}(t)$ if the failure times of type 1 components have a CDF $F_1(t) = 1 - e^{-t^2}$, and the failure times of type 2 components have a CDF $F_2(t) = 1 - e^{-t}$.

2.2. *Example*

Consider the system in Figure 1(b), consisting of 8 components of 3 types, $m = 8$ and $k = 3$. The notations A, B, C, D, E, F, G and H represent the specific components, and the notations 1, 2 and 3 represent the type of components. This system consist of three sub systems. The first subsystem is a parallel system consisting of components A and D, the second subsystem is a parallel system consisting of components B, E and G, and the third subsystem is a 2-out-of-3:G system consisting of components C, F and H.

The reliability of this system might be enhanced by a variety of swapping opportunities. In this example, we compare three swapping cases. In case 1, we assume that we are able to swap only type 2 components, in case 2, we assume that we are able to swap only type 3 components and in case 3, we assume that we are able to swap both type 2 and type 3 components, but a swap can only occur between components of the same type. We refer to original case without any swapping opportunity as case 0. With $m_1 = 3$, $m_2 = 3$ and $m_3 = 2$, $\Phi(l_1, l_2, l_3)$, $\Phi_{C_1}(l_1, l_2, l_3)$, $\Phi_{C_2}(l_1, l_2, l_3)$ and $\Phi_{C_3}(l_1, l_2, l_3)$ are given in Table 2. In this table we

l_1	l_2	l_3	Φ	Φ_{C_1}	Φ_{C_2}	Φ_{C_3}	l_1	l_2	l_3	Φ	Φ_{C_1}	Φ_{C_2}	Φ_{C_3}	l_1	l_2	l_3	Φ	Φ_{C_1}	Φ_{C_2}	Φ_{C_3}
0	2	2	1/3	1	1/3	1	2	1	1	2/9	2/3	4/9	1	3	1	0	1/3	1	1/3	1
0	3	1	1/2	1/2	1	1	2	1	2	5/9	1	5/9	1	3	1	1	2/3	1	1	1
0	3	2	1	1	1	1	2	2	0	1/3	2/3	1/3	2/3	3	1	2	1	1	1	1
1	1	2	2/9	2/3	2/9	2/3	2	2	1	1/2	5/6	7/9	1	3	2	0	2/3	1	2/3	1
1	2	1	2/9	2/3	4/9	1	2	2	2	7/9	1	7/9	1	3	2	1	5/6	1	1	1
1	2	2	5/9	1	5/9	1	2	3	0	2/3	2/3	2/3	2/3	3	2	2	1	1	1	1
1	3	0	1/3	1/3	1/3	1/3	2	3	1	5/6	5/6	1	1	3	3	0	1	1	1	1
1	3	1	2/3	2/3	1	1	2	3	2	1	1	1	1	3	3	1	1	1	1	1
1	3	2	1	1	1	1	3	0	1	1/2	1/2	1	1	3	3	2	1	1	1	1
2	0	2	1/3	1/3	1/3	1/3	3	0	2	1	1	1	1							

Table 2. $\Phi(l_1,l_2,l_3)$, $\Phi_{C_1}(l_1,l_2,l_3)$, $\Phi_{C_2}(l_1,l_2,l_3)$ and $\Phi_{C_3}(l_1,l_2,l_3)$ of system in Figure 1(b)

present only nonzero values. The reliability of this system can be calculated straightforwardly from this table by implementing equation (3) in case 0 and equation (5) in cases 1,2 and 3. Figure 2(b) presents the reliability of the system in each case, if we assume that the failure times of type 1 components have CDF $F_1(t) = 1-e^{-t^2}$, the failure times of type 2 components have CDF $F_2(t) = 1-e^{-t}$, and the failure times of type 3 components have CDF $F_3(t) = 1-e^{-t/2}$.

3. Component swapping with fixed penalty for system failure

Suppose that we have a system needs to continue functioning for a certain period of time $[0,\tau]$. If the system fails at any time t before the fixed time τ, a penalty cost needs to be paid. This penalty cost is fixed, independent of the failure time, denoted by c_p. If the system fails before τ, it can benefit from different swapping opportunities. An upfront cost need to be paid to enable each swapping opportunity. Let c_i denote the cost to enable swap i. We need to consider up front which opportunity of swapping cases will minimize the cost. We refer to case in which there is no swapping case as case 0. Let $C_0(\tau)$ denote the expected cost in case 0,

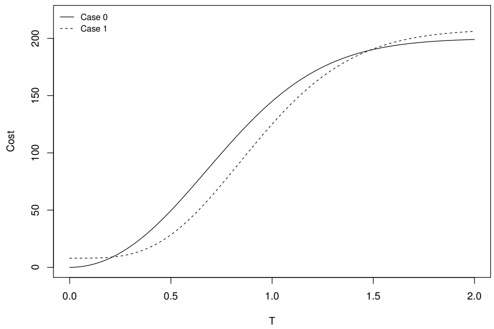
$$C_0(\tau) = c_p P(T_S < \tau) = c_p(1-R_s(\tau)) \tag{7}$$

Let $C_i(\tau)$ denote the expected cost in case i,

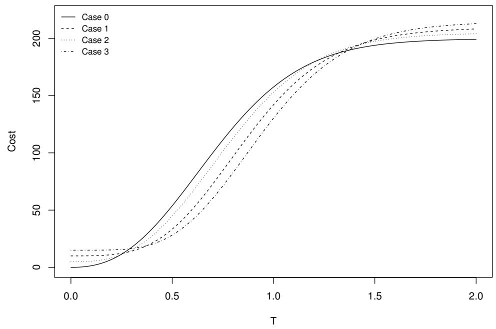
$$C_i(\tau) = c_i + c_p P(T_S^i < \tau) = c_i + c_p(1-R_{s,i}(\tau)) \tag{8}$$

3.1. *Example*

Consider the system in Figure 1(a) and we keep the same scenario for the swapping case and the failure time distribution as in example 2.1. We assume that the system needs to continue functioning for a period of time $[0,\tau]$. Assume that the penalty cost if the system fails down in the period $[0,\tau]$ is fixed and is $c_p = 200$, and the case 1 swapping cost is $c_1 = 8$. So, $C_0(\tau) = 200\left[1-\left[[F_1(\tau)][1-F_1(\tau)][1-[F_2(\tau)]^2]+[1-F_1(\tau)]^2\right]\right]$, and $C_1(\tau) = 8+200\left[1-\left[2[F_1(\tau)][1-F_1(\tau)][1-[F_2(\tau)]^2]+[1-F_1(\tau)]^2\right]\right]$. If τ = 1, the cost in case 0 would be 145.01 and in case 1 would be 125.08, which means that if τ = 1, it is good to take the opportunity of the swapping case. Figure 3(a) illustrates how the expected cost would change in case 0 and 1, depending on the value of τ. The opportunity of components swap will minimize the cost of the failure only if $0.21 < \tau < 1.48$ because if τ is small failure is unlikely and if τ is large failure is very likely even with a component swap.



(a) For the system in Figure 1(a)



(b) For the system in Figure 1(b)

Figure3. The expected cost with fixed penalty.

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3.2. *Example*

Consider the system in Figure 1(b) and we keep the same scenario for the swapping cases and the failure time distributions as in example 2.2. Assume that if the system fails before the fixed time τ, the penalty cost is fixed and is $c_p = 200$. The cost to enable swapping cases 1, 2 and 3 are $c_1 = 10$, $c_2 = 5$ and $c_3 = 15$, respectively. If we assume that $\tau = 1$, $C_0(1) = 157.53$, $C_1(1) = 142.00$, $C_2(1) = 152.98$ and $C_3(1) = 130.30$, which means that the opportunity of swapping case 3 should be taken to minimize the expected cost. We plot $C_0(\tau)$, $C_1(\tau)$, $C_2(\tau)$ and $C_3(\tau)$ as functions of τ in Figure 3(b). It is clear from this figure that for most values of τ it is either optional not to prepare for any swaps ($\tau < 0.24$ or $\tau > 1.37$), or to prepare for both swaps $(0.29 < \tau < 1.37)$, this is explained by the same reason as discussed in example 3.1.

4. Component swapping with time dependent penalty for system failure

In this section we assume that the penalty cost that needs to be paid if the system fails before a fixed time τ is dependent on the failure time. Let c represent the cost per unit of time if the system does not function in the period $[0,\tau]$. This system can benefit from different swapping opportunities if it fails at any time before τ. An upfront cost needs to be paid to enable each swapping opportunity, let c_i denote the upfront cost to be paid to enable swap i. In case 0, if the system fails at time $T_S \in [0,\tau]$, then the down time is $\tau - T_S$ and the down time cost is equal to $c(\tau - T_S)$. Let $C_0(\tau)$ denote the expected cost in case 0,

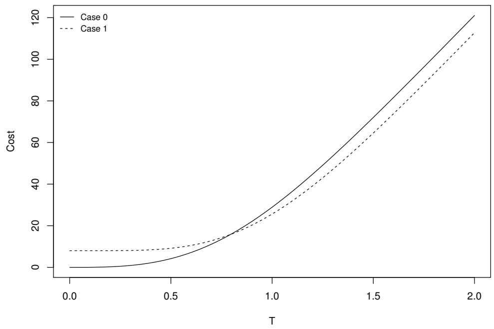
$$C_0(\tau) = cE(\tau - T_S) = c\left[\tau - \int_0^\tau R_s(t)dt\right] \tag{9}$$

Let $C_i(\tau)$ denote the expected cost in case i,

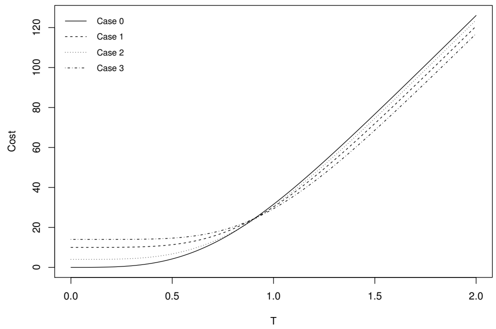
$$C_i(\tau) = c_i + cE(\tau - T_S^i) = c_i + c\left[\tau - \int_0^\tau R_{s,i}(t)dt\right] \tag{10}$$

4.1. *Example*

Consider the system in Figure 1(a) and, we keep the same scenario for the swapping case and the failure time distribution as in example 2.1. For the period of time $[0,\tau]$, we assume that the penalty failure cost per unit of time is $c = 100$. The swapping cost in case 1 is $c_1 = 8$. The expected cost in case 0 is $C_0(\tau) = 100\Bigg[\tau - \int_0^\tau \Big[[F_1(t)][1-F_1(t)][1-[F_2(t)]^2] + [1-F_1(t)]^2\Big]dt\Bigg]$, and in case 1 is $C_1(\tau) = 8 + 100\Bigg[\tau - \int_0^\tau \Big[2[F_1(t)][1-F_1(t)][1-[F_2(t)]^2] + [1-F_1(t)]^2\Big]dt\Bigg]$. If $\tau = 1$, then the expected cost in case 0 is 28.91 and in case 1 is 25.63, which means that it is better to take the opportunity of case 1 swap. Figure 4(a) illustrates how the expected cost would change in case 0 and 1, depending on the value of τ. If τ is small, the system is unlikely to fail,



(a) For the system in Figure 1(a)



(b) For the system in Figure 1(b)

Figure4. The expected cost with with time dependent penalty.

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while for large τ the system is likely to fail but the swapping case delayed the failure time, leading to lower penalty costs.

4.2. *Example*

Consider the system in Figure 1(b) and we keep the same scenario for the swapping cases and the failure time distributions as in example 2.2. Let $c = 100$ and the cost to enable swapping cases 1,2 and 3 are $c_1 = 10$, $c_2 = 4$ and $c_3 = 14$, respectively. If $\tau = 2$, the expected cost in cases 0, 1, 2 and 3, would be 126.06, 120.72, 123.77 and 116.96, respectively. Thus, to minimize the cost we should prepare for swapping case 3. Figure 4(b) illustrates how the expected cost would change in cases 0, 1, 2 and 3, depending on the value of τ.

5. Concluding remarks

In this paper we considered cost effective component swapping to increase system reliability. We discussed the cost of component swapping with fixed penalty for system failure and with time dependent penalty. Considering a single period $[0, \tau]$ is often practically relevant and convenient for the analysis, as shown in this paper. A topic for further research is consideration over an infinite, or undefined, length of time, for example by the use of renewal theory (Tijms, 1990). Further interesting topics for future research are different cost structures and consideration of swapping, component standby, spares and maintenance activities.

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