Editorial

How pleasing to read some poetry in December's issue! Mathematicians are famed for their all-round talents and many excel at physical and creative activities. Now Charles Evans and Alan Champneys very ably demonstrate considerable prowess with their poetic skills. Who could guess that a conversation with a stranger on a bus would subsequently reveal strong links between poetry, chaotic maps and prime numbers? Perhaps this is the true beauty of our subject: it crops up everywhere.

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A short while ago, I had the good fortune to visit Stratford-upon-Avon, the home of our national poet William Shakespeare, to attend a superb performance of *Julius Caesar*. The acting and staging were fabulous, though the most important element was his brilliant use of language. I also performed in his comedies *Twelfth Night* and *A Midsummer Night's Dream* with a local amateur dramatics society and learned to appreciate the clever sonnets and iambic pentameters that generate appealing rhythms. Poetry comprises a variety of structural elements and so offers a rich source of material for mathematical investigation. Good starting points are an article written by JoAnne Growney [1] and some recently published mathematical haiku [2].

Astute readers might have spotted my lack of a control group in the first paragraph, where I fail to acknowledge that many non-mathematicians are also multi-talented. This form of publication bias is common in practice and distorts reality. Curricula vitae mention applicants' successes yet omit their failures, while social media devotees might post only their most attractive selfies. Lottery adverts on television proudly display the winners yet ignore all the losers, while a popular radio station broadcasts ecstatic claims by victorious competition entrants about how easy it is to win!

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Of greater consequence is the publication of scientific research. Statistical analyses that generate negative results could remain unpublished because either the investigators withhold the findings or the journals are unimpressed. Funnel plots and forest plots are able to identify such publication bias for systematic reviews and meta-analyses, while Bayes' theorem makes fair inference possible when bias exists. This formula can be expressed conveniently as

$$\frac{P\,(A \mid B)}{P\,(A' \mid B)} = \frac{P\,(B \mid A)}{P\,(B \mid A')} \times \frac{P\,(A)}{P\,(A')}$$

or 'posterior odds equal Bayes factor times prior odds'.

One of its most important applications is in criminal law, where the formula enables us to evaluate the odds in favour of guilt A given evidence B, or the posterior probability $P\,(A \mid B)$ if preferred. Indeed, the logarithm of the first term on the right-hand side of this expression

$$w = \log \frac{P\,(B \mid A)}{P\,(B \mid A')}$$

represents the weight of evidence in favour of guilt rather than innocence, according to Good [3]. The prior odds can be assigned a default value of one to convey equal likelihood of guilt and innocence before evidence is considered. To ensure a presumption of innocence until proven guilty, the posterior odds should be sufficiently large in order for the jury to return a guilty verdict. As with all binary classifications under uncertainty though, there is a balance between sensitivity and specificity. This approach also suggests that the extent of any penalty (fine, service, incarceration, etc.) imposed by a judge should be a continuous function of the posterior odds, though this proposal is far too radical for now. It would effectively tax some innocent people to avoid severe penalties for an unfortunate few, and would doubtless be unpopular.

Although explicit use of these formulae is discouraged in courts of law, skilful barristers are crucially acquainted with conditional probability. Ignoring this logic has led to tragic miscarriages of justice and some fascinating paradoxes that were exposed in previous issues of *Mathematics Today*: the interrogator's fallacy [4]; the prosecutor's fallacy and the defender's fallacy [5]; the jury observation fallacy [6]; and definitions of reasonable doubt [7]. The Royal Statistical Society has led the way in improving numeracy in matters of litigation and recently published an excellent introductory guide to statistics and the law [8].

An interesting civil case came to my attention a few years ago. This concerns a national institution that advances large amounts of funding to service providers. It subsequently sued several of these for breach of contract and used sampling methods to ascertain the amounts owing as reimbursement for incorrect service provision. One of these lawsuits involved skilfully selecting a large, representative sample of transactions and laboriously evaluating the discrepancy (about £40k) between the alleged costs and services provided for those transactions. This was then scaled up to estimate the total discrepancy in the population of all transactions (about £678k) under this contract.

Criminal law requires evidential proof beyond reasonable doubt to find a defendant guilty, in which case such evidence would exist only for the sum of £40k here. However, civil law merely requires enough evidence for a balance of probabilities, which is a more precise definition but is harder to implement in this context.

Does this imply that the national institution's claim for reimbursement of £678k is valid? Although this is the most likely amount owed by the defendant, there is a slight possibility that all but 6% of this sum was rightly spent by this service provider. If such an extrapolated claim were valid, then how large a sample would be needed to provide a reliable estimate? A small sample might generate the same estimate as a large sample but would be much less reliable.

Perhaps the balance of probabilities should decide each transaction independently. This would again suggest £40k in total, even though the most likely amount falsely claimed is about

£678k. Maybe the answer lies with Bayesian analysis, which generates conclusions such as: 'the total discrepancy in the population is at least £538k with probability 0.95'. This issue was conveniently sidestepped as these lawsuits were eventually dismissed because the lengthy contracts turned out not to be legally binding!

I encountered another interesting sampling problem last year, courtesy of fellow villager and IMA member Graham Unwin. It relates to online quality control, which generally involves occasionally selecting items from a production line and ascertaining whether they are functional or defective, in which case they are recycled. This is a very common procedure that applies to such diverse items as smart devices, woolly jumpers and chocolate truffles. With such binary observations, the binomial distribution determines suitable warning and action limits for monitoring and recalibrating the production process.

The current scenario differs because it relates to detecting particular types of fault during the continuous manufacture of glass. These occur randomly and rarely, at a rate of μ faults per unit area. A highly sensitive test assesses about 1% of the glass in each production run.

The problem is as follows. Suppose that a particular run produces an area A of glass and contains an unknown number N of faults. A sample with area a is taken from this run and found to contain n faults. If there are no faults in the sample, what is the probability that there are no faults in the run?

The challenge requires us to evaluate the conditional probability that $N=0$ given that $n=0$. The appropriate model is a Poisson point process, which gives

$$P(N=0 \mid n=0)=\exp\{-(A-a)\mu\},$$

so the required probability p depends on the difference between the areas of population and sample, rather than the ratio as might be expected. To illustrate this numerically, suppose that the areas are $A=100\,\mathrm{m}^2$ and $a=1\,\mathrm{m}^2$. If faults occur at a rate of 1 per $100\,\mathrm{m}^2$ on average, then $p\approx 0.37$. If this rate reduces to 1 per $1{,}000\,\mathrm{m}^2$ on average, then $p\approx 0.91$. These probabilities hardly change if no sampling occurs, so why bother? Then again, if we never did any sampling then how could we estimate the average number of faults per unit area?

That is quite enough from me. I hope that 2019 provides you with health, wealth, happiness and some exciting new challenges!

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