

Karen Uhlenbeck wins Abel Prize

Karen Uhlenbeck is the 2019 recipient of the Abel Prize. The citation awards the prize ‘for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics.’ She was one of the founders of geometric analysis: how did this come about?

In the mid 60s, Uhlenbeck (then Karen Keskulla) began graduate school at Brandeis University, with Palais as her adviser; this naturally led her to the calculus of variations. A fundamental question in this field (among Hilbert’s famous problems from 1904) is to prove the existence of *critical points* of certain energies (area, elastic energy, etc.). A critical point is an equilibrium configuration with respect to the given energy; the equilibrium condition is expressed mathematically via the Euler–Lagrange equation.

A classical example is a geodesic on a Riemannian manifold and, in order to prove the existence of geodesics, it proves effective to consider the energy as a functional on the (infinite-dimensional) ‘space of curves’. In this abstract setting, R. Palais and S. Smale identified a condition (that nowadays bears their names) on functionals defined on (infinite-dimensional) Hilbert manifolds. The validity of this condition guarantees that one can detect a critical point by considering a sequence of ‘almost solutions’, i.e. points in the Hilbert space that fail to satisfy the Euler–Lagrange equation by an error that converges to zero along the sequence, and then by taking their limit, which provides the critical point.

This approach led to the successful proof of existence of geodesics thanks to the fact that the Palais–Smale condition is true for the relevant functional. However, many functionals of interest fail to satisfy the condition: this is the case of *harmonic maps*, a higher-dimensional generalisation of geodesics. Given M and N closed Riemannian manifolds, for $u : M \rightarrow N$ the Dirichlet energy of u is

$$F(u) = \int_M |\nabla u|^2.$$

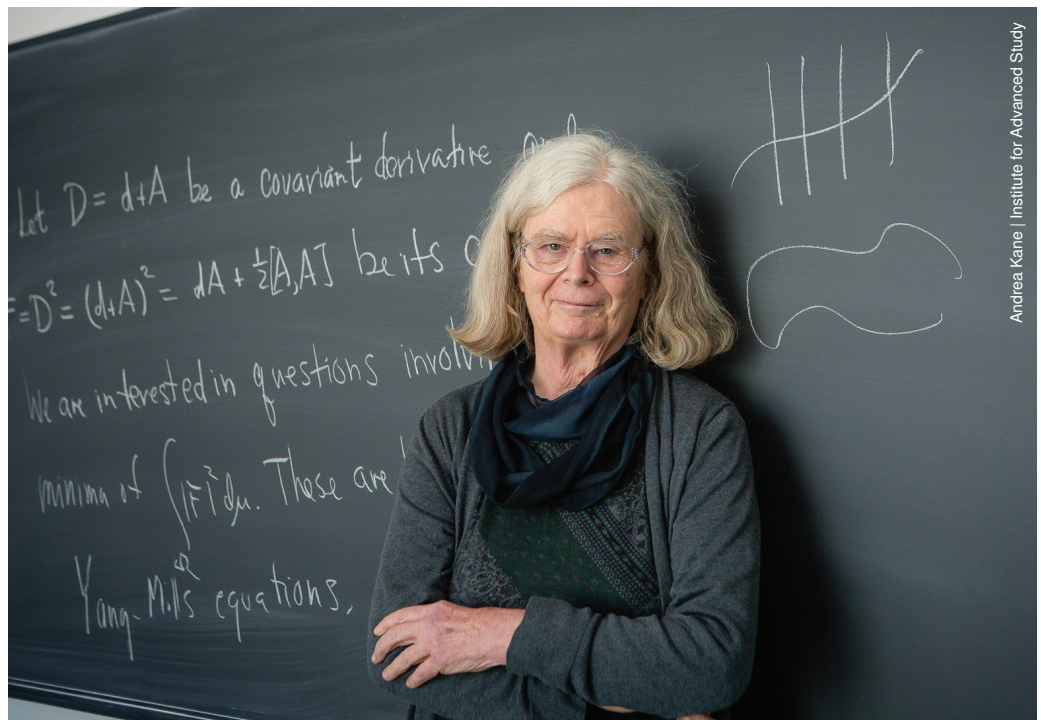
A harmonic map is a critical point of the Dirichlet energy. If we imagine M made of rubber, looking for a harmonic map means that we are seeking a way to situate M inside N so that it sits in equilibrium, i.e. it does not snap into different positions.

Already in the case $\dim M = 2$, the problem posed considerable challenges: the Palais–Smale condition fails for F , moreover, the Euler–Lagrange equation does not enter the framework of classical elliptic partial differential equations.

In the 70s, while at Urbana–Champaign, Uhlenbeck devised, in a ground-breaking paper [1] co-authored with J. Sacks, an alternative method to detect harmonic maps (in the case $\dim M = 2$) by approximation: the key was to consider a sequence of energies

$$F_\alpha(u) = \int_M (1 + |\nabla u|^2)^\alpha,$$

for $\alpha > 1$; these energies satisfy the Palais–Smale condition and, as $\alpha \rightarrow 1$, they approximate F . The Sacks–Uhlenbeck paper constructed a sequence u_α of critical points of F_α and



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carried out a careful analysis of the limiting behaviour of u_α as $\alpha \rightarrow 1$, finding that u_α converge smoothly to a limit u away from a finite collection of points $\{p_1, \dots, p_N\}$. Moreover, the limit u extends smoothly across the points p_j , providing a harmonic map $u : M \rightarrow N$ (possibly constant).

The paper then successfully analyses the behaviour of u_α around the points p_j showing that suitable rescalings of u_α around these points converge to (at least one) non-constant harmonic map from S^2 to N . The ‘bubbling phenomenon’ had been discovered: this is a singular behaviour in which harmonic spheres (‘bubbles’) are lost in the limit. In order for this to happen, it is necessary that the target N allows the formation of such bubbles, which is a topological condition on the manifold.

Uhlenbeck realised that bubbling is the reason behind the failure of the (analytic) Palais–Smale condition on F . The

geometric picture of bubbling obtained in the paper relies on a key analytic result: if in a disk the energies $F_\alpha(u_\alpha)$ stay below a certain threshold ε_0 , then one has smooth convergence of u_α on a slightly smaller disk; the quantity ε_0 depends only on the geometric data M and N and not on the size of the disk. Uhlenbeck realised that in the low-energy regime (energy below ε_0) the Euler–Lagrange equation behaves like a classical elliptic PDE, while in the high-energy regime (energy above ε_0) the failure of classical PDE estimates is caused by the possibility of bubbling.

Bubbling has since then appeared in a variety of PDE and geometric contexts. Beyond the strict mathematical content of the paper, Uhlenbeck’s insight had, for the first time, provided beautiful bridges between topology, differential geometry and analysis of PDEs: this would evolve in the field that is nowadays known as *geometric analysis*.

Uhlenbeck’s ideas in mathematics have made a major impact on theoretical physics mainly thanks to her work in gauge theory, in which she became interested in the 80s. Gauge theory lies at the basis of the Standard Model of particle physics and deals with critical points of the so-called Yang–Mills energy. This functional is defined on the space of connections on a vector bundle; the energy is the L^2 -norm of the curvature of a connection and critical points are called instantons. The functional and the geometric data are invariant under a certain group action (gauge-invariance): an effect of this invariance (which is a completely new aspect when compared to the harmonic map setting) is that the associated Euler–Lagrange system may look very different depending on the choice of gauge (which is reminiscent of a choice of coordinates).

Uhlenbeck introduced a new analytical viewpoint and proved the fundamental fact that in a suitable gauge (the so-called Coulomb gauge) the Euler–Lagrange system becomes elliptic. This provided the basis for her next results [2,3]: a compactness theorem for connections with curvatures bounded in L^p , and her famous ‘singularity removability theorem’: the latter proves that an instanton that is well-defined in the punctured ball $B^4 \setminus \{0\}$ and has finite Yang–Mills energy, can be smoothly extended across the point. These results triggered a Yang–Mills analogue of the Sacks–Uhlenbeck bubbling analysis, as well as major subsequent developments of the theory; moreover, they have been key for the applications of gauge theory to geometry and topology of 4-manifolds.

Uhlenbeck’s mathematics is incredibly broad and undeniably beautiful. But her influence on the mathematical community is just as important. She is the first woman to win the Abel Prize since its foundation in 2002, and has been breaking the glass ceiling for years: way back in 1986, she became the first woman mathematician elected to the National Academy of Sciences.

But she is more than a role model for women mathematicians: she is a leading advocate. Her undergraduate and graduate study took place at a time when there were very few women indeed in mathematics; she worked in a non-tenure track lectureship for three years because no institution wanted to offer both her and her husband tenure-track posts. However, she was not discouraged [4]:

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There was blatant, overt discouragement, but also subtle encouragement. A lot of people appreciated good students, male or female, and I was a very good student. I liked doing what I wasn’t supposed to do, it was a sort of legitimate rebellion. There were no expectations because we were women, so anything we did well was considered successful.

Following on from her NAS election, she began to work explicitly for women in mathematics. In 1991 she was one of the co-founders of the Park City Mathematics Institute at the Institute for Advanced Study at Princeton, an achievement of which she is especially proud [5]. She organised a mentoring program in which, over a two-week period,

women participate in seminars, working problem groups, and mentoring and networking sessions, and have the opportunity to meet and converse with mathematicians in residence at the Institute for Advanced Study. In 1993, Uhlenbeck co-founded the Women and Mathematics (WAM) Program at Princeton’s Institute for Advanced Study ‘with the mission to recruit and retain more women in mathematics’ (www.math.ias.edu/wam).

The progress that has been made for women in mathematics is astonishing – colleagues from physics and engineering often ask how we manage to recruit such a gender-balanced undergraduate population – but we are nowhere near a fully balanced community. Role models like Karen Uhlenbeck are still needed.

We leave you with a quote from Karen herself about the difficult state of being a role model [4]:

I am aware of the fact that I am a role model for young women in mathematics, and that’s partly what I’m here for. It’s hard to be a role model, however, because what you really need to do is show students how imperfect people can be and still succeed. Everyone knows that if people are smart, funny, pretty or well-dressed they will succeed. But it’s also possible to succeed with all of your imperfections. It took me a long time to realise this in my own life. In this respect, being a role model is a very unglamorous position, showing people all your bad sides. I may be a wonderful mathematician and famous because of it, but I’m also very human.

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