The Unit Re-balancing Problem

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Abstract

We describe the problem of optimally re-balancing several military units distributed over a large geographic area of locally independent domains (such as islands). Each unit consists of three components: the number of people, their armor, and their equipment. For each of the three components, a nonlinear function is introduced that converts their numerical status (a real number between 0 and 1) into an assessment number. This allows comparing the components between different units as well as with other components of the same unit. Over time, units become unbalanced so that they are too costly and too inefficient at the same time. They need to be re-balanced by moving components between different units. The desired goal is to have units that are equally well equipped at the lowest possible cost. On a secondary level, the cost for the re-balancing should also be as low as possible. We describe a mixed-integer nonlinear programming formulation for this problem. This model formulation describes the potential movement of components between units as a multi-commodity flow at minimum cost. A special feature of this model is the inclusion of table-based piecewise-linear functions. In this way, nonlinear functions can be re-formulated by introducing additional binary variables and constraints to fit into the framework of mixed-integer linear programming. We present numerical solutions for a set of test instances and a bi-criteria objective function, demonstrating the trade-off between cost and efficiency.

1. Introduction

In recent years many countries tried to re-organize the structure of their army and security forces (army, police, fire departments, first aids, etc.) in order to have better readiness and to be more efficient. At the same time this re-organization has to lead to new structures that can be used with the same or even lower budget.

The basic idea is to re-balance the force, manpower, and equipment by moving some supporting capabilities from the reserve component to the active component. The aim was to bring the reserve component force structure into closer alignment with that of the active component. The purpose of this re-balancing is to reduce the size of the institutional forces - the sustaining base - to an unprecedented low percentage of the total forces.

Re-balancing type problems received a lot of attention from researchers from various scientific fields. Many applications from an economic perspective have already occurred which used the re-balancing problem in order to construct hedged portfolios or find the optimal portfolio under certain restrictions (see Takahashi & Tsuzuki 2017, Wagalath 2014).

There are a few applications of re-balancing in logistics such as the Bike Re-balancing

Problem (see Schuijbroek, Hampshire & van Hoeve 2017, Cruz *et al.* 2017) in which the main issue is to re-distribute the bikes with the objective of minimizing the total cost. In the Bike Re-balancing Problem the main idea is to use repositioning which is usually done by capacitated vehicles based in a central point which pick up bicycles from stations where the level of occupation is too high and deliver them to stations where the level is currently too low (see Dell'Amico *et al.* 2014). A similar problem exists for the re-location of vehicles at car-sharing companies, see Bruglieria, Colorni & Lue 2014, Boyaci, Zografos & Geroliminis 2015.

In computer science, re-balancing is used to solve problems concerning several aspects such as search trees, assignment problems (i.e., a set of jobs needs to be assigned) or load re-balancing problems (see Sen, Tarjan & Kim 2016, Aggarwal, Motwani & Zhu 2006). In Belikovetsky & Tamir (2016) the authors study the load re-balancing problem as a game. The load re-balancing problem refers to the case where a set of tasks should be assigned to a set of machines. Each task is associated with a processing time and the goal is to balance the load on the existing machines.

In Sathe & Miller-Hooks (2005) the authors focus on locating and re-locating a fleet of response units (from army, police, etc.) in a specific transportation network to optimally cover a number of facilities. They present the problem as a mixed-integer linear program and develop a heuristic algorithm. Another well known application of the re-location (re-balancing) problem is the one that deals with emergency vehicles (see Liu *et al.* 2013, Paul, Lunday & Nurre 2017).

The use of the term re-balancing as we want to understand it has to do with the dynamics. We use this term instead of re-location because in the case where we have some army units we want to re-balance their critical different components (force, manpower, equipment) from one to another unit in order to increase the efficiency. In this framework the term re-balancing is more appropriate than re-location since re-location deals mainly with finding the best location for each unit. Our approach, for army units, deals with all the components that an army unit has and characterizes them.

The remainder of this paper is organized as follows. In Section 2, we present the mathematical model with variables, an objective function, and restrictions. In Section 3, we review the data which inspired this paper and show how the current-day situation can be improved with the help of our model. Section 4 contains computational results with which the performance of different numerical solvers is analysed. Finally in Section 5, we provide some concluding remarks and directions for possible future research.

2. Model

2.1. *Sets and Parameters*

An organization such as an army or police force consists of several units. These units are numbered in the set $\mathcal{U} := \{1, 2, \ldots, U\}$, index u. Every unit can be described by various characteristics which are included in the set $\mathcal{C}$, index c. In this paper, we want to keep our focus on army units. Hereinafter, let $\mathcal{C} := \{\mathtt{ar}, \mathtt{pe}, \mathtt{eq}\}$ (for armor, personal, equipment). The total area on which the units are distributed, e.g., a country, is divided into regions, e.g., counties or islands. The set of regions is defined as $\mathcal{R} := \{1, 2, \ldots, R\}$, index r.

Before the optimization, the status quo of all units is notated in a matrix $(p_{c,u})_{c,u}$ with entries in $[0, 1]$. Every entry of this matrix can be attached to a status: To determine the status of each characteristic in a unit, we apply continuous piecewise linear functions to $p_{c,u}$. This way, statuses can take any value in $[0, S_{\max}]$, 0 being the lowest and $S_{\max}$ being the highest possible status. Breakpoints of each piecewise linear function are given in the matrix $(g_{c,i})_{c,i}$, $i \in \{1, 2, \ldots, S_{\max}\}$ which contains the values of $p_{c,u}$ that correspond to an integer status value. It follows that $g_{c,S_{\max}} = 1$ for all $c \in \mathcal{C}$. We also assume that $p_{c,u} = 0$ results in a corresponding status of 0. Additionally, $x^p_u \in \{0, 1\}$ tells us if a unit u is open or not.

The cost to maintain a unit depends on its overall status, which is defined as the maximum status among its characteristics. In an analogous manner, the efficiency value of a unit is determined by its minimum status among its characteristics. Cost and efficiency of a unit are determined similar to its status. The vectors g^k and g^v contain the cost and efficiency values of the integer statuses $1, 2, \ldots, S_{\max}$ and serve as breakpoints of the respective continuous piecewise linear functions. The statuses $s^p_{c,u}$, cost k^p_u and efficiency v^p_u of a unit prior to the re-balancing can be calculated beforehand. As bookkeeping parameters we introduce

$$K^p := \sum_{u=1}^{U} k^p_u \in \mathbb{R}^+, \quad V^p := \sum_{u=1}^{U} v^p_u \in \mathbb{R}^+,$$

which reflect the cost and efficiency of all units "before", respectively.

Every unit has coordinates X_u, Y_u and is located in a region r_u with $r_u \in \mathcal{R}$. For some regions, in particular capitals and smaller islands, it is desired to specify efficiency values in the set $\mathcal{B}$, index b with $0 \leq b \leq S_{\max}$ together with lower and upper bounds on the number of units in this region that have at least this particular efficiency value, specified as $\underline{n}_r(b)$, $\overline{n}_r(b) \in \mathbb{N}$ with $0 \leq \underline{n}_r(b) \leq \overline{n}_r(b) \leq U$. Furthermore, shipping costs vary between regions. Let γ_{r_1,r_2} be the shipping cost factor between regions r_1 and r_2. Different characteristics also have different shipping cost factors κ_c. Total shipping costs are bounded by $\sigma_{\max} \geq 0$. The Euclidean distance between two units is given by

$$d : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+, (u_1, u_2) \mapsto \sqrt{(X_{u_2} - X_{u_1})^2 + (Y_{u_2} - Y_{u_1})^2}.$$

2.2. *Variables*

During the re-balancing, the amount of characteristic c that is transferred from unit u_1 to unit u_2 is stored in $f_{c,u_1,u_2} \in [0, 1]$.

After the re-balancing, $q_{c,u} \in [0, 1]$ notates the value of characteristic c in unit u. In $x^q_u \in \{0, 1\}$, we can see if unit u is still open or not.

The bookkeeping variables $K^q \in \mathbb{R}^+$ and $V^q \in \mathbb{R}^+$ represent the cost and efficiency of all units "after", respectively.

The status of the three characteristics in every unit after the re-balancing is stored in $s^q_{c,u} \in [0, S_{\max}]$. To model the continuous piecewise linear functions necessary to determine a status, the incremental method by Markowitz & Manne (1957) is used with auxiliary variables $w_{c,u,i}$ and $\delta_{c,u,i}$, $i \in \{1, \ldots, S_{\max}\}$. To determine the maximum status $s^{\max,q}_u$ and minimum status $s^{\min,q}_u$ of each unit, the following auxiliary variables are added: $z^{\max}_{u,1}$, $z^{\max}_{u,2}$, $z^{\min}_{u,1}$, $z^{\min}_{u,2} \in \{0, 1\}$; $\lambda^{\max}_u$, $\lambda^{\min}_u \in [0, S_{\max}]$. The cost k^q_u and efficiency v^q_u of each unit is computed similarly to $s^q_{c,u}$ with auxiliary variables $w^k_{u,i}$, $\delta^k_{u,i}$, $w^v_{u,i}$, $\delta^v_{u,i} \in [0, S_{\max}]$, $i \in \{1, \ldots, S_{\max}\}$. Finally, $\theta_{u,b} \in \{0, 1\}$ tells us if the minimum status of a unit u is higher than b.

2.3. *Objective*

The goal is to maximize the total efficiency after re-balancing the units, additionally minimizing the required shipping costs:

$$\max \; V^q - \varepsilon\sigma$$

with

$$\sigma = \sum_{\substack{c\in C,\\ u,u_1\in U}} \kappa_c \cdot \gamma_{r_u,r_{u_1}} \cdot d(u,u_1) \cdot f_{c,u,u1},$$

and ε being a sufficiently small value (we use $\varepsilon := 10^{-6}$).

2.4. *Constraints*

The total efficiency "before" and "after" is computed by adding up the efficiency of the individual units:

$$V^p = \sum_{u\in\mathcal{U}} v_u^p, \quad V^q = \sum_{u\in\mathcal{U}} v_u^q.$$

The total costs "before" and "after" are computed similarly:

$$K^p = \sum_{u\in\mathcal{U}} k_u^p, \quad K^q = \sum_{u\in\mathcal{U}} k_u^q.$$

The budget is limited. The re-balanced units ("after") must not be more expensive than "before":

$$K^q \le K^p.$$

A closed unit has to be emptied completely ("before" and "after"):

$$p_{c,u} \le x_u^p, \quad q_{c,u} \le x_u^q, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}.$$

An open unit must have a minimum amount of any characteristic ("before" and "after"):

$$0.01 \cdot x_u^p \le p_{c,u}, \quad 0.01 \cdot x_u^q \le q_{c,u}, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}.$$

The existing units can be distributed (cannibalized) to other units:

$$p_{c,u} = \sum_{u_1\in U} f_{c,u,u_1}, \quad q_{c,u} = \sum_{u_0\in U} f_{c,u_0,u}, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}.$$

The status "after" is modeled as a continuous piecewise linear function of $q_{c,u}$:

$$w_{c,u,i} \ge \delta_{c,u,i}, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}, i \in \{1,\dots,S_{\max}\},$$

$$\delta_{c,u,i} \ge w_{c,u,i+1}, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}, i \in \{1,\dots,S_{\max}-1\},$$

$$q_{c,u} = g_{c,1}\delta_{c,u,1} + \sum_{i=2}^{S_{\max}} (g_{c,i} - g_{c,i-1})\delta_{c,u,i}, \quad s_{c,u}^q = \sum_{i=1}^{S_{\max}} \delta_{c,u,i}, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}.$$

To find the maximum status of a unit among its three characteristics "after", the maximum of the armor and personal statuses is identified first:

$$\lambda_u^{\max} \ge s_{\mathtt{ar},u}^q, \quad \lambda_u^{\max} \ge s_{\mathtt{pe},u}^q, \quad \forall\, u \in \mathcal{U},$$

$$\lambda_u^{\max} \le s_{\mathtt{ar},u}^q + S_{\max} z_{u,1}^{\max}, \quad \lambda_u^{\max} \le s_{\mathtt{pe},u}^q + S_{\max}(1 - z_{u,1}^{\max}), \quad \forall\, u \in \mathcal{U}.$$

This candidate for the maximum status is thereupon compared with $s_{\mathtt{ch},u}^q$ to find the "true" maximum $s_u^{\max,q}$:

$$s_u^{\max,q} \ge \lambda_u^{\max}, \quad s_u^{\max,q} \ge s_{\mathtt{eq},u}^q, \quad \forall\, u \in \mathcal{U},$$

$$s_u^{\max,q} \le \lambda_u^{\max} + S_{\max} z_{u,2}^{\max}, \quad s_u^{\max,q} \le s_{\mathtt{eq},u}^q + S_{\max}(1 - z_{u,2}^{\max}), \quad \forall\, u \in \mathcal{U}.$$

The minimum status $s_u^{\max,q}$ of a unit is determined similarly:

$$\lambda_u^{\min} \le s_{\mathtt{ar},u}^q, \quad \lambda_u^{\min} \le s_{\mathtt{pe},u}^q, \quad \forall\, u \in \mathcal{U},$$

$$\lambda_u^{\min} \geq s_{\texttt{ar},u}^q - S_{\max} z_{u,1}^{\min}, \quad \lambda_u^{\min} \geq s_{\texttt{pe},u}^q - S_{\max}(1 - z_{u,1}^{\min}), \quad \forall\, u \in \mathcal{U},$$

$$s_u^{\min,q} \leq \lambda_u^{\min}, \quad s_u^{\min,q} \leq s_{\texttt{eq},u}^q, \quad \forall\, u \in \mathcal{U},$$

$$s_u^{\min,q} \geq \lambda_u^{\min} - S_{\max} z_{u,2}^{\min}, \quad s_u^{\min,q} \geq s_{\texttt{eq},u}^q - S_{\max}(1 - z_{u,2}^{\min}), \quad \forall\, u \in \mathcal{U}.$$

The cost and efficiency of a unit are continuous piecewise linear functions of $s_u^{\max,q}$ and $s_u^{\min,q}$, respectively:

$$w_{u,i}^k \geq \delta_{u,i}^k, \quad \forall\, u \in \mathcal{U}, i \in \{1, \ldots, S_{\max}\},$$

$$\delta_{u,i}^k \geq w_{u,i+1}^k, \quad \forall\, u \in \mathcal{U}, i \in \{1, \ldots, S_{\max} - 1\},$$

$$s_u^{\max,q} = \sum_{i=1}^{S_{\max}} \delta_{u,i}^k, \quad k_u^q = g_1^k \delta_{u,1}^k + \sum_{i=2}^{S_{\max}} (g_i^k - g_{i-1}^k)\delta_{u,i}^k, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C},$$

$$w_{u,i}^v \geq \delta_{u,i}^v, \quad \forall\, u \in \mathcal{U}, i \in \{1, \ldots, S_{\max}\},$$

$$\delta_{u,i}^v \geq w_{u,i+1}^v, \quad \forall\, u \in \mathcal{U}, i \in \{1, \ldots, S_{\max} - 1\},$$

$$s_u^{\min,q} = \sum_{i=1}^{S_{\max}} \delta_{u,i}^v, \quad v_u^q = g_1^v \delta_{u,1}^v + \sum_{i=2}^{S_{\max}} (g_i^v - g_{i-1}^v)\delta_{u,i}^v, \quad \forall\, u \in \mathcal{U}, c \in \mathcal{C}.$$

The minimum status of a unit may or may not exceed a given level:

$$S_{\max}\theta_{u,b} \leq S_{\max} + s_u^{\min,q} - b, \quad s_u^{\min,q} - b \leq S_{\max}\theta_{u,b} - \varepsilon, \quad \forall\, u \in \mathcal{U}, b \in \mathcal{B}.$$

Every region must contain a certain number of units with their lowest status between given levels:

$$\underline{n}_r(b) \leq \sum_{\substack{u \in U: \\ r_u = r}} \theta_{u,b} \leq \overline{n}_r(b), \quad \forall\, r \in \mathcal{R}, b \in \mathcal{B}.$$

Shipping costs are not allowed to exceed $\sigma_{\max}$:

$$\sum_{\substack{c \in C, \\ u, u_1 \in U}} \kappa_c \cdot \gamma_{r_u, r_{u_1}} \cdot d(u, u_1) \cdot f_{c,u,u1} \leq \sigma_{\max}.$$

3. Input Data

Goal of this paper is the re-balancing of $U = 75$ army units which are distributed over $R = 12$ regions. For a given status quo, each characteristic of every unit can be assigned to a status $\in [0, 5]$. The breakpoints for the piecewise linear functions of armor, personal and equipment are defined by the test functions $x \mapsto x^2$, $x \mapsto \sqrt{x}$ and $x \mapsto x$, respectively. For example, a square function used for the `pe` characteristic would impose a quadratic benefit from adding more personal to a unit, which resembles to Lanchester's square law, see Lanchester (1956). The breakpoints are given as follows:

$$(g_{c,i})_{c,i} := \begin{array}{c} \\ \texttt{ar} \\ \texttt{pe} \\ \texttt{eq} \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.45 & 0.63 & 0.77 & 0.89 & 1.00 \\ 0.04 & 0.16 & 0.36 & 0.64 & 1.00 \\ 0.20 & 0.40 & 0.60 & 0.80 & 1.00 \end{pmatrix} \end{array}.$$

Cost and efficiency coefficients are chosen as follows:

$$g^k = (0.27, 0.51, 0.68, 0.81, 0.94), \quad g^v = (0.16, 0.38, 0.49, 0.76, 0.89).$$

r	b	$\underline{n}_r(b)$	$\overline{n}_r(b)$	r	b	$\underline{n}_r(b)$	$\overline{n}_r(b)$	r	b	$\underline{n}_r(b)$	$\overline{n}_r(b)$
3	3	4	4	9	3	8	8	11	3	4	4
3	4	2	2	9	4	3	3	11	4	2	2
7	3	8	8	10	3	4	4	12	3	4	4
7	4	3	3	10	4	2	2	12	4	2	2

TABLE 1. Regional restrictions for a 75-unit instance.

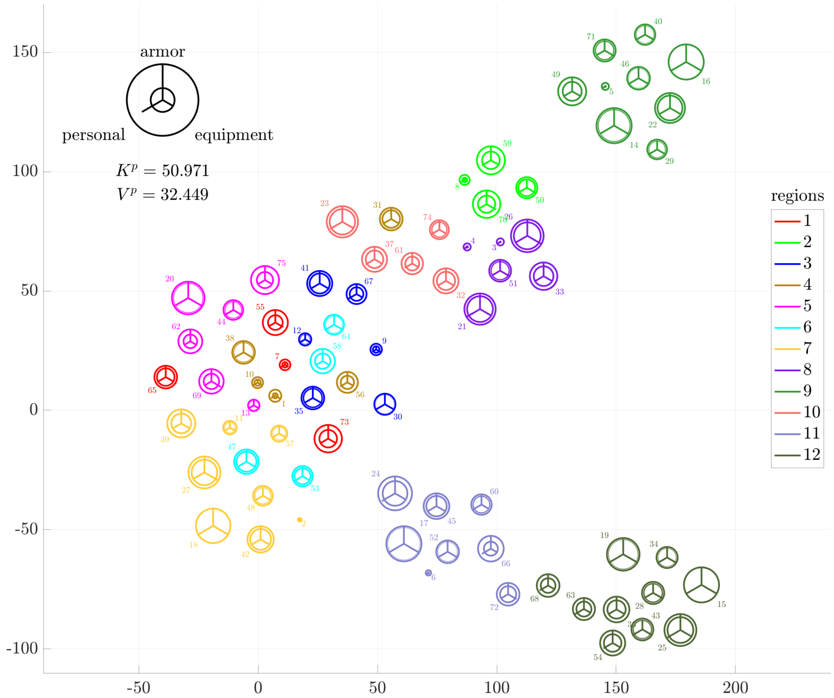

FIGURE 1. Location and strength of units prior to optimization.

The regional restrictions are listed in Table 1. Furthermore, let $\kappa_{\mathtt{ar}} = 0.9$, $\kappa_{\mathtt{pe}} = 0.3$, $\kappa_{\mathtt{eq}} = 0.8$. An overview of the location and initial strength of each unit is given in Figure 1.

Before the re-balancing, the total efficiency equals $V^p = 32.449$ with total costs of $K^p = 50.971$. Afterwards, the efficiency is increased to $V^q = 37.279$ (14.9% increase) while total costs have fallen to $K^q = 48.754$. Shipping costs for the re-balancing operations sum up to $\sigma = 750.834$. As a result of the re-balancing process, six units were closed. The effects can be seen in Figure 2. The movements of each characteristic between the regions can be visualized in alluvial flow diagrams (see Carmeli 2018). Exemplarily, the shifts of personal are shown in Figure 3.

4. Computational Results

After formulating the continuous model as a mixed-integer linear programming problem as described in Section 2, numerical solvers can be used to find feasible solutions and — in the best case — verify that the provided solution is optimal. For our computations

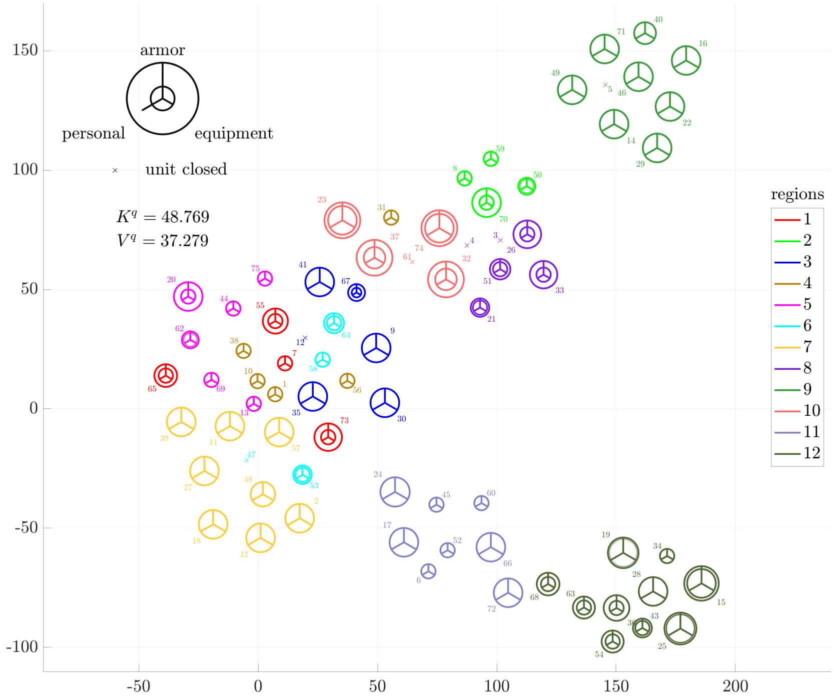

FIGURE 2. Location and strength of units after optimization.

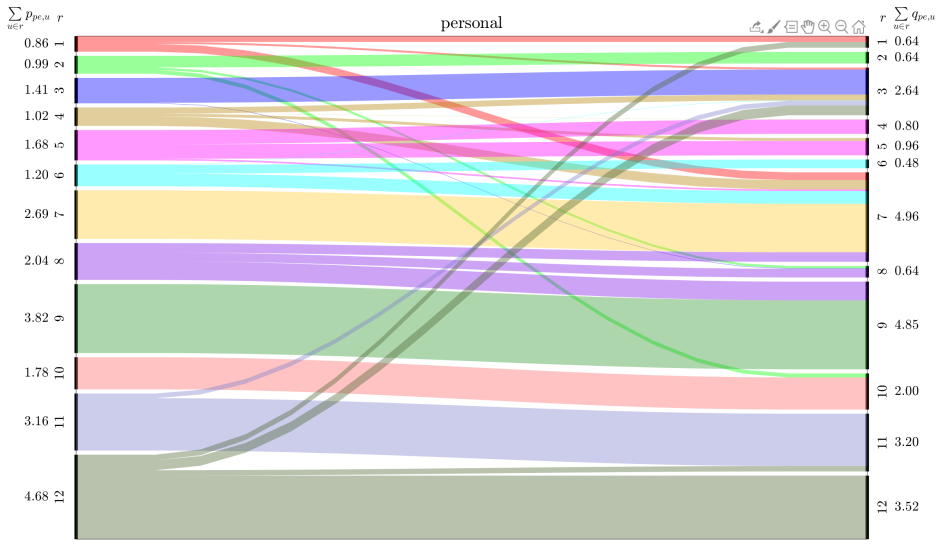

FIGURE 3. Alluvial flow diagram displaying shifts of personal between regions.

we used an Apple MacBook Pro laptop running an Intel Core i7 at 3.10 GHz clock speed and 16 GB RAM. All instances were provided for the algebraic modeling language AMPL (version 20190927). The following solvers were available: CPLEX 12.9, Gurobi 8.1, Xpress 8.6.

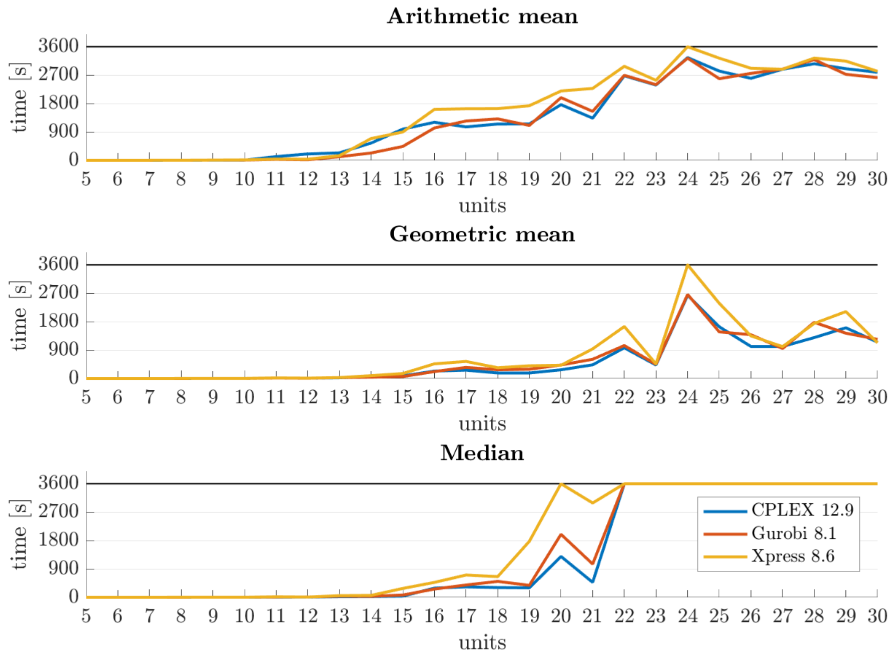

FIGURE 4. Average solution times of the three solvers depending of the number of units.

4.1. *Comparison of Solvers*

To identify the best among the three solver for instances of our model, we compared their performance on a variety of instances. For a fixed maximum status ($S_{\max} = 5$) and number of regions ($R = 9$), we varied the number of units between 5 and 30, generating 20 instances each. Coordinates were randomly assigned on a 100×100 square area. Regions were determined by dividing the area into thirds horizontally and vertically. Shipping costs were generated so that $\gamma_{r_1,r_2} = \gamma_{r_2,r_1}$ for all $r_1, r_2 \in \mathcal{R}$ and $\kappa_{\mathtt{ar}} \geq \kappa_{\mathtt{eq}} \geq \kappa_{\mathtt{pe}}$. Regional restrictions were not enforced. For this sample, Gurobi returned the best results. With Gurobi, instances were solved fastest or, when reaching the maximum computation time of one hour, left the smallest relative gaps between the best feasible solution found and the best upper bound. Average solution times for each solver can be seen in Figure 4. In total, Gurobi needed 684,422 seconds to solve all instances with the remaining gaps summing up to 147%. CPLEX needed 704,981 seconds and left a total sum of gaps of 207%. Solving all instances with Xpress took the longest time with 809,152 seconds and summed gaps of 263%. Consequently, we used Gurobi for all following computations. It is noteworthy that although many instances exceed the maximum computation time of one hour, Gurobi left gaps no bigger than 6.4%. In most cases, solutions with a remaining gap of $< 1\%$ could be found in under a minute.

4.2. *Trade-off between Budget And Efficiency*

In our model, we demand that the total costs of all units K^q after the re-balancing cannot exceed the total costs K^p beforehand. This constraint could either be loosened (e.g., by allowing a maximum cost increase of 10%) or tightened (e.g., by additionally forcing a 20% reduction of the total costs). With allowing a cost increase, one would

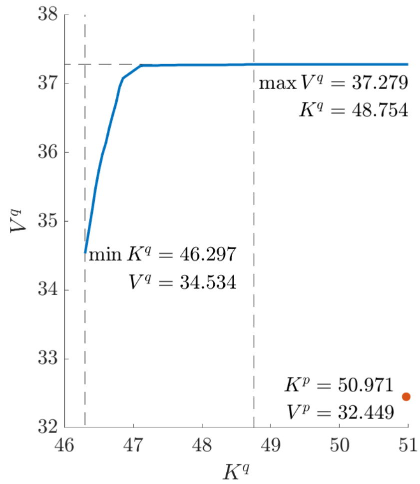

FIGURE 5. Trade-off between budget and efficiency for a 75-unit instance.

hope for a corresponding increase in total efficiency. On the other hand, reducing the total costs will probably lessen total efficiency. To check the effects of increasing or decreasing the maximum total costs, we reverted to the instance solved in Section 3. As we can see, the solution which maximizes efficiency includes total costs after of $K^q = 48.754$ which are well below the total costs before ($K^p = 50.971$). Raising the barrier for the maximum total costs would have no effect on the optimal solution. Therefore, the remaining question is how low the total costs can be pushed and which impact this has on the total efficiency. For this instance, the lowest possible costs (assuming no destruction of material or dismissal of personal) sum up to $K^q = 46.297$ (5.0% less compared to the "highest efficiency" solution). However, efficiency also dropped to $V^q = 34.534$ (7.4% decrease). To identify the behavior of costs and efficiency between these two extremes, instances were solved where the maximum costs were restricted to a value in the interval $[46.30, 48.75]$. In these computations, shipping costs were ignored to speed up the process. In total, 51 instances were solved within a time restriction of one hour each, leaving a maximum gap of 0.22%. Total computation time was 109,626 seconds. The results are shown in Figure 5.

5. Discussion, Conclusions, and Outlook

We gave a nonlinear mixed-integer model formulation for the re-balancing of resources or assets for security services. The resources or assets are moved from one unit to another. The model can be re-formulated as a mixed-integer linear program, so that it is solveable with standard numerical solvers for this problem class. It turns out that it is possible to achieve an increasing active operating strength and more efficiency with the same operational budget.

The study employed anonymized data from army units. The results indicate that re-balancing the resources has an extremely positive influence on the efficiency of the units. At the same time the total operational cost can be reduced. The way of organizing the units after the re-balancing procedure creates a force that is more effective for today's tasks.

Future research in this topic could focus on the trade-off between budget and efficiency. As the example in Figure 5 showed, total costs can be reduced significantly if one allows a minor reduction of efficiency. Consequently, the objective could be replaced by

$$\max\ \alpha V^q - \beta K^q$$

with $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Shipping costs would be disregarded in this formulation as a reduction of long-term costs could be considered more important than the one-time shipping expenses.

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