Srinivasa Ramanujan (1887–1920) The Centenary of a Remarkable Mathematician

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This month marks the centenary of the death of one of the most remarkable mathematicians of the 20th century. The enigmatic Indian mathematician Srinivasa Ramanujan was perhaps one of the most original mathematicians of all time. In a career that only lasted around ten years, he produced hundreds of highly innovative results in several areas of pure mathematics, particularly number theory and analysis. After his death, his notebooks and unpublished results inspired decades of research by succeeding mathematicians, the impact of which is still being felt in mathematics today. What follows is based in large part on the work of these scholars, particularly G.H. Hardy [1,2], George Andrews [3], Bruce Berndt [3–5] and Ramanujan's biographer, Robert Kanigel [6].

Srinivasa Ramanujan was born to a modest Brahmin family on 22 December 1887 in the town of Erode in Tamil Nadu, southern India. Due to his father's heavy work schedule, the boy formed a close relationship with his mother, and it was from her that he acquired his religious beliefs and adherence to specific customs, particularly his strict vegetarianism. Performing well at primary school, he passed exams in English, Tamil, geography and arithmetic at the age of 10 with the best scores in his school district. After beginning secondary level mathematics, by his early teens he was investigating and discovering his own independent results. For example, by the age of 15, having been taught the method for solving cubic equations, he had created his own algorithm to solve the quartic.

The biggest influence on his mathematical development appears to have been his acquisition, at the age of 16, of a copy of *A Synopsis of Elementary Results in Pure and Applied Mathematics* by a certain G.S. Carr [7]. This was a large compendium of hundreds of mathematical formulae and theorems, listed thematically with little to no commentary or motivation. Although the book did contain brief derivations of some of the results, such demonstrations were minimal and the overall content was terse and dry. Nevertheless, its content appears to have had a profound effect on Ramanujan, stimulating a fascination with formulae and symbolic manipulation that never left him, and influencing his preferred style of mathematical presentation.

Upon graduation from secondary school in 1904, he was awarded a scholarship to study at the Government College in Kumbakonam, also in Tamil Nadu. However, his obsession with his own mathematical research resulted in his failing important exams in most of his other subjects of study and consequently losing his scholarship. His health was also proving problematic, with absences from classes caused by illness occurring with some frequency. Later enrolling at Pachaiyappa's College, Madras (now called Chennai), he failed to obtain a degree in 1907 for essentially the same reasons as before. Forced to leave college at the age of 20, with no qualifications, poor health and poorer career prospects, the next few years were of considerable hardship for Ramanujan as he lived in poverty while trying to forge ahead with his research in mathematics.

As early as 1904, Ramanujan had begun to produce mathematical research of substantial sophistication. Of course, having no contact with any active mathematical researchers and being essentially self-taught, he had no knowledge of contemporary research topics, so would largely pursue his own ideas, often inspired by formulae and techniques from Carr's *Synopsis*. He was particularly fascinated by infinite series and processes. An early example of a problem he solved was to find the value of

Figure 1: Srinivasa Ramanujan (1887–1920)

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}.$$

Ramanujan's solution noted that, since $n\sqrt{1+(n+1)(n+3)} = n(n+2)$, if we let $f(n) = n(n+2)$, then

$$\begin{aligned} f(n) &= n\sqrt{1+f(n+1)} \\ &= n\sqrt{1+(n+1)\sqrt{1+f(n+2)}} \\ &= n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+f(n+3)}}}, \end{aligned}$$

and so on, so that, ultimately,

$$n(n+2) = n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\cdots}}}}.$$

Letting $n=1$ then revealed that

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}} = 3.$$

In July 1909, Ramanujan was married to Janaki Ammal in an arranged marriage that took place when Janaki was just 10 years old. Although such a practice was not unusual at the time, Ramanujan's wife did not actually live with him for the first three years of their marriage, moving in with him and his mother in 1912. Meanwhile his reputation as a mathematician began, very slowly, to grow. Following his introduction in 1910 to V. Ramaswamy Aiyer, the founder of the Indian Mathematical Society, Ramanujan began to publish his results in the Society's *Journal*. His first paper, 'Some properties of Bernoulli's numbers' [8], stemmed from research undertaken in his teens, when he had discovered and developed the Bernoulli numbers in complete ignorance of any prior research on the subject.

But he continued to struggle to find gainful employment to support his family. After combinations of temporary clerical work and private tutoring, he managed to secure a position as a clerk in the Madras Port Trust. His growing circle of mathematical friends in the Madras area became convinced that his work should be brought to the attention of mathematicians in Britain. Early attempts were disappointing, however. A letter to M.J.M. Hill, professor of mathematics at University College London, received a faintly encouraging, but largely patronising, response, while letters to Cambridge mathematicians E.W. Hobson and H.F. Baker received no reply. It was then, in January 1913, that Ramanujan wrote one of the most famous letters in the history of mathematics.

Its recipient was G.H. Hardy, a lecturer at Trinity College, Cambridge, who was one of Britain's foremost pure mathematicians. On reading the multi-page letter crammed with dozens of intricate formulae and theorems from Ramanujan's notebooks, Hardy's first reaction could have been to dismiss everything as the work of a fraud or a crank. But a close examination of its content by Hardy and his Cambridge colleague J.E. Littlewood revealed a host of amazing results. These Hardy divided into three categories. Firstly, there were theorems that, unbeknown to Ramanujan, were already known, such as the integral formula:

$$\int_0^a \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{2}\pi^{1/2} - \cfrac{\mathrm{e}^{-a^2}}{2a + \cfrac{1}{a + \cfrac{2}{2a + \cfrac{3}{a + \cfrac{4}{2a + \cdots}}}}}.$$

Secondly, there were results that, while new, were interesting rather than important, for example:

$$\int_0^\infty \frac{\cos \pi x}{\Gamma^2(\alpha + x)\Gamma^2(\alpha - x)}\,\mathrm{d}x = \frac{1}{4\Gamma(2\alpha - 1)\Gamma^2(\alpha)},$$

where $\alpha > 1/2$ and $\Gamma(z)$ is the gamma function. And finally, there were entirely original results that were simply astonishing, such as:

$$\cfrac{1}{1 + \cfrac{\mathrm{e}^{-2\pi}}{1 + \cfrac{\mathrm{e}^{-4\pi}}{1 + \cdots}}} = \left[\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}\right] \mathrm{e}^{\frac{2}{5}\pi}.$$

These formulae, Hardy later wrote [4, part II, p. 103],

> … defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

It quickly became apparent that Ramanujan was a mathematician of exceptional ability, and Hardy soon began to consider ways in which he could be brought to Britain to receive formal training and an advanced degree from Cambridge. Ramanujan, however, was initially unwilling to consider leaving his wife and family; so, as a short-term measure, his mathematical friends in India applied, successfully, for a temporary research scholarship on his behalf at the University of Madras, which allowed him to concentrate on his mathematical research full time while supporting his family financially. But there was still the matter of persuading him to make the long sea journey to England. This he refused to do for a number of reasons, some of them religious and some related to family obligations. But after a year of wrangling, these obstacles were gradually overcome and, with his parents' eventual consent, he finally agreed to go, setting sail for England on 17 March 1914.

On his arrival in Cambridge the following month, he immediately set to work with Hardy and Littlewood. Not surprisingly, however, they soon found that their mathematical methodologies were profoundly different. Hardy and Littlewood, being strict analysts, were insistent on absolute rigour and formal proofs, while Ramanujan was content to rely on intuition and inductive experimentation. Indeed, as Littlewood later wrote [9, p.88],

> He was not interested in rigour … [and] the clear-cut idea of what is *meant* by a proof … he perhaps did not possess at all.

Equally inevitably, given Ramanujan's erratic higher education, there were also substantial gaps in his mathematical knowledge. It was a source of constant surprise to Hardy that, despite Ramanujan's expertise in elliptic functions (almost certainly self-taught), he had no experience at all in the theory of complex functions. Hardy thus sought to remedy the defects in Ramanujan's formal training without dampening his enthusiasm and wild imagination. It proved to be a challenge.

Adapting to life in Britain was equally challenging for Ramanujan. In addition to the colder and damper climate, the practice of vegetarianism was not nearly as easy in early 20th century Cambridge as it is today. He cooked his own simple meals in his room in Trinity College, and thus did not eat with the rest of the college community as would usually have been expected. The outbreak of the First World War in August 1914, not long after his arrival in Britain, made things harder still, as rationing made it increasingly difficult to maintain a healthy diet with the limited variety of produce on offer. The war also resulted in many of Cambridge's best mathematicians, including Littlewood, being called away for war service, further adding to Ramanujan's sense of isolation.

Nevertheless, Ramanujan's mathematical output was rapid. Within months he was publishing papers in British journals, including a 60-page paper [10] in the *Proceedings of the London Mathematical Society* in 1915. This was on a subject of his own creation called 'highly composite numbers'. Whereas number theorists are often largely concerned with properties and attributes of prime numbers, Ramanujan considered integers 'whose number of divisors exceeds that of all its predecessors' and deduced a startling array of results about them. It was largely on the basis of this work that, in March 1916, he was awarded the degree of Bachelor of Science by Research (renamed PhD in 1920) by the University of Cambridge.

Figure 2: G.H. Hardy (1877–1947)

Before long, Ramanujan and Hardy were collaborating on joint research and much of 1916 was spent working on their most famous collaboration: their joint paper on partitions [11]. The partition number $p(n)$ represents the number of ways that a positive integer n can be written as a sum of positive integers where the order of addition does not matter. So for instance, the integer $n = 4$ can be written in five different ways, namely,

$$\begin{aligned} 4 &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1, \end{aligned}$$

meaning that $p(4) = 5$. But the question quickly becomes more complicated. For example, $p(15) = 176$, and $p(34) = 12\,310$. What then is $p(100)$ or $p(200)$?

After months of effort, Ramanujan and Hardy produced an answer to this question in the form of one of the most staggering formulas in the whole of mathematics. If

$$A_q(n) = \sum_{p=1}^{q} \omega_{p,q}\, \mathrm{e}^{-2np\pi \mathrm{i}/q}$$

and

$$\phi(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}} \left[\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\exp(a\lambda_z/q)}{\lambda_z} \right) \right]_{z=n}$$

where $p, q \in \mathbb{Z}^+$, $(p,q) = 1$, $a = \pi\sqrt{2/3}$, $\lambda_z = \sqrt{z - 1/24}$ and $\omega_{p,q}$ is a $24q$-th root of unity, then, for some $\alpha \in \mathbb{Z}^+$,

$$p(n) = \sum_{q=1}^{[\alpha\sqrt{n}]} A_q(n)\phi(n) + O(n^{-1/4}).$$

It is really remarkable to get a result with an error of $O(n^{-\frac{1}{4}})$ which is as good as that with any higher powers of n in this problem. Let c_s be the s-series which is $O(s)$ and $\Omega(1)$. Suppose now that s is about $\beta\sqrt{n}/\log n$ where $\beta = 4\pi\sqrt{2/3}$. Then

$$O \sum_{O(s)}^{O(s)} \frac{|c_s|\sqrt{s}}{sn} \mathrm{e}^{(\pi/s)\sqrt{(2n/3)}} = O(s\sqrt{s}n^{-\frac{3}{4}}) = O(\log n)^{-\frac{3}{2}}.$$

It would therefore follow from your arguments that the error by taking about $\beta\sqrt{n}/\log n$ terms is $O(\log n)^{-\frac{3}{2}}$. Again suppose that s is about $\alpha\sqrt{n}/\log n$ where $\alpha = (4\pi/5)\sqrt{2/3}$, then

$$\frac{|c_s|\sqrt{s}}{sn} \mathrm{e}^{(\pi/s)\sqrt{(2n/3)}} = \Omega\left(\frac{n^{\frac{1}{4}}}{\sqrt{s}}\right) = \Omega\left(\sqrt{\log n}\right).$$

It therefore appears that, in order that $p(n)$ may be the nearest integer to the approximate sum, s need not be taken beyond $\beta\sqrt{n}/\log n$ and cannot be taken below $\alpha\sqrt{n}/\log n \ldots$

Figure 3: Extract from a postcard sent by Ramanujan to Hardy as their work on partitions neared completion [5, p. 141].

They then proceeded to show that their formula was capable of producing results of unprecedented accuracy. For $n = 100$, it gave an output of 190 569 291.996, with the actual partition number being 190 569 292. For $n = 200$, the result was just as dramatic: their formula gave 3 972 999 029 388.004, with an error of just 0.004. Hardy and Ramanujan's formula displayed a breathtaking blend of mathematical ideas and influences. Fundamentally, it was a fusion of Ramanujan's dazzling powers of formulaic intuition with Hardy's mastery of the tools of analytic function theory. But without the intuitive genius of Ramanujan, Hardy would never have formulated such an astonishing result; and without Hardy, Ramanujan would never have been able to prove it. Littlewood said [9, p. 90],

> We owe the theorem, to a singularly happy collaboration of two men, of quite unlike gifts, in which each contributed the best, most characteristic, and most fortunate work that was in him.

Ramanujan's health had never been strong, but by May 1917 he was seriously ill, with his condition no doubt exacerbated by the British weather and the difficulty of maintaining adequate nutrition in wartime. He was diagnosed with tuberculosis and severe vitamin deficiency, although recent analysis has concluded that he may have been suffering from hepatic amoebiasis, a complication arising from previous attacks of dysentery. He spent much of the year in various nursing homes, but his mathematical output, although reduced, remained as remarkable as ever. His spirits were raised by his election to membership of the London Mathematical Society in December 1917, followed by fellowships of the Royal Society in May 1918 and Trinity College in October 1918. The war had enforced a prolonged stay in England, but by November 1918, his health had improved sufficiently for Hardy to write about a return to his homeland [5, p. 200]:

> He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is. His natural simplicity and modesty has never been affected in the least by success – indeed all that is wanted is to get him to realise that he really is a success.

On 27 February 1919, he embarked for India, arriving in Kumbakonam two weeks later, but his health deteriorated again despite medical treatment. He died on 26 April 1920 at the age of 32.

Ramanujan was described as being enthusiastic and eager, with a good-natured personality, although somewhat shy and quiet in official settings. Not particularly introspective, he was never able to give a completely coherent account of how he came up with his ideas – indeed, although his religious beliefs were later downplayed by Hardy (an atheist), there is evidence that Ramanujan believed that some form of divine inspiration was involved. In any case, he seems to have been quite modest about his own abilities and scrupulously keen to acknowledge help from any other sources. Littlewood famously remarked [2, p. xxxv] that 'every positive integer was one of his personal friends', and one of the best known stories told by Hardy appears to corroborate that opinion [2, p. xxxv]:

> I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number $(7 \cdot 13 \cdot 19)$ seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways.'

With regard to his mathematics, its prime characteristic is its overwhelming wealth of algebraic formulae and vast computational complexity. Ramanujan was gifted with a power of calculation and symbolic dexterity unavailable to most mathematicians prior to the computer age. He also had an uncanny ability to spot patterns that nobody knew existed. For example, from the list of partition numbers from 1 to 200, he deduced a number of attractive, but hitherto unknown, congruences, including

$$p(5m+4) \equiv 0 \mod 5$$

and

$$p(25m+24) \equiv 0 \mod 25.$$

Most mathematicians would be satisfied with the mere discovery of relationships such as these, but in order to prove them Ramanujan was led to an even more stunning result

$$p(4) + p(9)x + p(14)x^2 + \cdots = 5\frac{\{(1-x^5)(1-x^{10})(1-x^{15})\dots\}^5}{\{(1-x)(1-x^2)(1-x^3)\cdots\}^6},$$

which is one of the most beautiful formulae he ever produced.

His ability to conjure up a myriad of bizarre yet almost supernaturally accurate approximations was another overwhelming feature of his mathematics. From his work on elliptic and modular functions came irrational expressions surprisingly close to integer values, such as

$$e^{\pi\sqrt{58}} = 24\,591\,257\,751.999\,999\,822\dots$$

It also yielded a series of tremendously accurate approximations to π, for example,

$$\frac{1}{2\pi\sqrt{2}} \approx \frac{1103}{99^2},$$

which is correct to 8 decimal places. But like so much of Ramanujan's mathematics, there is far more profound detail lying beneath the surface, since this approximation is merely the first term of an exact identity

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26\,390k)}{(k!)^4 396^{4k}},$$

which is itself intimately related to his earlier value of $e^{\pi\sqrt{58}}$, since 26 390 is a multiple of 58, while $396^{4k} = (4^2 \times 99^2)^{2k}$, and

$$e^{\pi\sqrt{58}} = 396^4 - 104.000\,000\,177\dots$$

In the 100 years since his death, Ramanujan's mathematics has provided a constant source of inspiration and wonder for mathematicians. His work with Hardy on the partition function introduced a powerful new technique, subsequently developed by Hardy and Littlewood, now known as the 'circle method', which remains a valuable tool in additive number theory. The Hardy–Ramanujan partition formula itself was refined and improved by Hans Rademacher in the 1930s and, as well as its utility in mathematics, now serves as a useful function in superstring theory in physics and the study of phase transitions in chemistry. Ramanujan's manuscript notebooks – containing well over 3000 formulae and theorems – have been analysed intensively, with their contents being proved over the decades by subsequent generations of mathematicians. The discovery by George Andrews of a further 'lost notebook' in 1976 revealed a further 600 results. The mathematics contained within these notebooks, together with Ramanujan's published papers and innovations such as the Ramanujan theta function, Ramanujan primes and mock theta functions, have opened up new areas of mathematics and will no doubt continue to inspire and stimulate new mathematical ideas for many years to come.

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