What’s Knot to Love?

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Watch out! Your shoelaces are undone! ... At least that’s what a knot theorist would tell you. Mathematically, your laces are only a knot if they are fused at the end, forming a closed loop. This may seem pretty pedantic, but it is a distinction that opens the door to some wonderful results and applications.

In the 19th century, renowned physicist Lord Kelvin had the idea that all atoms were knots, with the different twists and turns giving each atom a unique set of properties. It is a strangely appealing concept. Those captivated by this idea set about trying to list all the distinct types of knots, with the hope of constructing a complete table of elements [1].

However, before this task was completed, Kelvin’s theory was proved incorrect. The physicists and chemists dropped knot theory entirely. To them, it was an irrelevant piece of mathematics without any real-world applications. No one could have predicted the impact that knot theory has on today’s society.

It is a relief then that pure mathematicians, as always, upon hearing about the lack of real application, grew giddy with excitement. Thus, modern knot theory was born.

So, let us start as they did, with a simple question: How do we tell two knots apart?

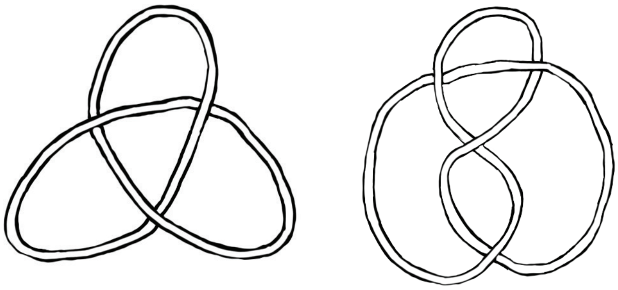

Figure 1: Is a trefoil knot equivalent to a figure of eight knot?

You can try it yourself. Get a piece of string or even an extension cable. Tie it up, and then fuse the ends together. See if you can transform the trefoil knot in Figure 1 into the other one. It is a finicky job, isn’t it? Eventually, you will conclude that these knots are in fact different after all. But, how can we make this process of wrestling with an extension cord into something mathematically rigorous?

Enter Kurt Reidemeister, who came up with three distinct ways of transforming a knot [2]. Instead of stumbling in the dark, mathematicians now had a way to communicate in the language of knots. If no combination of Reidemeister moves can turn one knot into the other, then we can be sure that they are distinct.

The three moves are quite self-explanatory and sound a bit like a mathematical dance: a twist, a poke and a slide (see Figure 2). These make up the basic things you can do to a diagram of a knot to try and transform it. This way we can see if it is distinct or another knot in disguise.

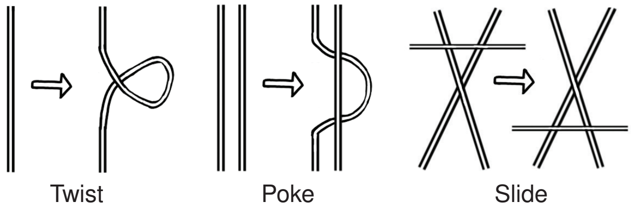

Figure 2: The three Reidemeister moves.

This notion of transforming is known as ambient isotopy, which plays an important role in the genre of maths called topology, of which knot theory is a branch.

Is this enough to tell any two knots apart? How can we really know that we have exhausted every combination of Reidemeister moves? What if there is some trick that completely untangles the knot, and we have just missed it?

Take this next example in Figure 3.

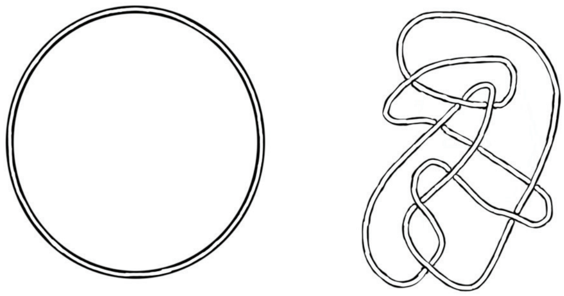

Figure 3: Is the unknot (trivial knot) equivalent to the tangle [2] on the right? Knot so easy this time.

This is where Reidemeister moves just don’t cut it. You could try to do the dance for hours on end and still get nowhere. In fact, there is a combination of moves that completely unravels that complicated knot in Figure 3 into nothing more than a simple loop. They are the same knot.

It turns out that Reidemeister moves are only the foundation of knot theory. The first real tool to solve our problem comes in the form of a very neat and visually pleasing piece of mathematics: tricolourability.

For a knot diagram to be tricolourable:

(i) Every section or strand of the knot must be filled with three different colours.

(ii) At each crossing, the three strands must be all the same colour or all different colours.

For instance, the unknot is not tricolourable because there is only one strand to colour, violating the first condition.

The second rule can be represented in equation form. Take the crossing shown in Figure 4.

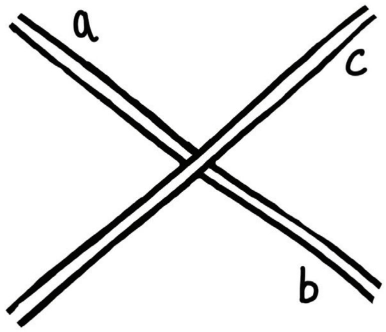

Figure 4: The standard crossing labelling process.

Then, the second condition can be written as:

$$a + b \equiv 2c \pmod 3 \text{ with } a, b, c \in \{0, 1, 2\}.$$

These numbers correspond to the colours we can fill the strands with, say {red, green, blue}.

You can try out any combination of the three labels, both allowed and forbidden to see that this equation gives an algebraic way of stating how the strands must be coloured.

This concept is wildly important thanks to our good friend Reidemeister, because his moves are preserved under tricolouring. The proof of this is shown in Figure 5. Usefully, this means that no matter what moves you make on a knot, it will stay tricolourable or stay not tricolourable.

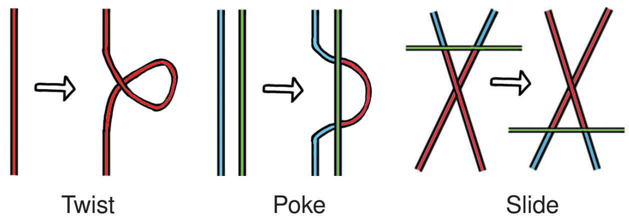

Figure 5: The Reidemeister moves preserve colourability.

Tricolourability is known as a knot invariant: a property that does not change under ambient isotopy, or under any moves you can make. The one exception is taking a pair of scissors to the knot. This is a move strictly forbidden in topology and severely frowned upon in the extension cord aisle of B&Q.

Knot invariance is the key to answering our original question. It means for sure that if one knot is tricolourable and the other is not (see Figure 6, for example), then these are different and there is no way to transform one into the other.

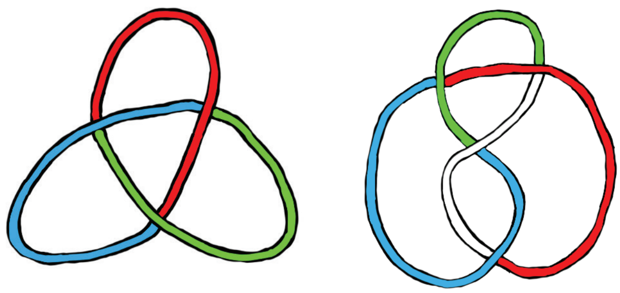

Figure 6: We can now show that these two knots are not equivalent. The trefoil is tricolourable, but the figure of eight is not.

Now we are getting somewhere. We can separate our knots into two different categories, tricolourable and not. However, we are not done yet. Take the figure of eight knot and the unknot. Both cannot be tricoloured but are clearly different.

It turns out that tricolourability is just a subset of a larger concept known as prime-colourability. Yes, it is exactly what you are thinking. Instead of three colours, we can use any other prime number, meaning instead of two different categories to put knots in, we will have many.

The same rules apply, just generalised. Our equation becomes:

$$a + b \equiv 2c \pmod p,$$

with $a, b, c \in \{0, 1, 2, \ldots, p-1\}$ and $p \geq 3$ and prime.

Now, before you get out your infinitely many and distinct coloured pens, it will be useful to look at another invariant: the knot determinant.

Using a knot diagram, it is possible to construct a matrix $\mathbf{M}$ that depends on each of the crossings and strands. The theorem we are going to demonstrate is simple to state:

A knot is p-colourable if and only if $p \mid \det(\mathbf{M})$.

Simple, yet you can see how useful it is, and much more mathematically rigorous than 3 felt tip pens and a dream.

Rather than read a set of instructions on how to build the matrix $\mathbf{M}$, let us consider the basic example of the trefoil knot, and hopefully come to the answer, which we have already confirmed.

First, we label each crossing and each strand as in Figure 7.

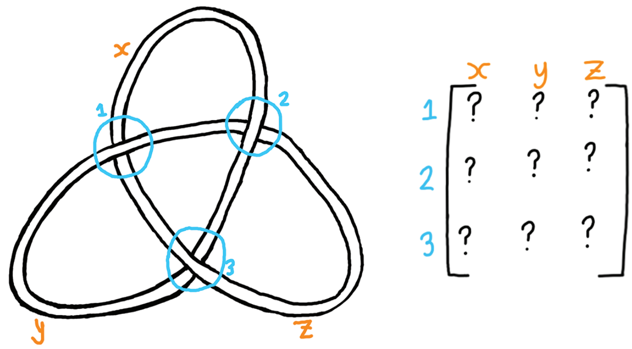

Figure 7: The first step in creating our knot matrix.

The crossings will become the row headings and the strands will become the column headings for our matrix.

The creation of the knot matrix is intrinsically linked with our equation $a + b \equiv 2c \pmod p$, which can be rearranged to $0 \equiv 2c - a - b \pmod p$. Notice the coefficients of the three letters corresponding to the strands at any crossing. In Figure 4, we saw that c represented the overcrossing, while a and b represented the two undercrossings.

To fill in the matrix, all we must do at each crossing is to identify which strand is the overcrossing and which two are the undercrossings. Due to the coefficients of our equation, in the matrix we enter 2 for the strand overcrossing, and -1 for the strands undercrossing.

For example, at crossing 1, strand y is the overcrossing, with x and z being undercrossings. Therefore, row one of our matrix will read $\begin{pmatrix} -1 & 2 & -1 \end{pmatrix}$. Filling in the rest gives us our completed matrix, shown in Figure 8. The next step is to cancel out one of the rows and one of the columns. It does not matter which one, just pick your least favourite. In this case, we will get a 2×2 matrix for which we can find the determinant.

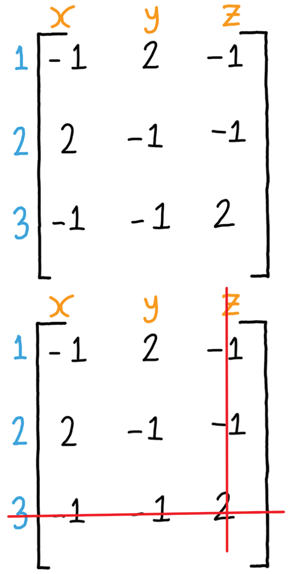

Figure 8: Completed matrix

As we can see, the determinant of the new matrix is ± 3, no matter which row and column you take out.

By our simple theorem we know that if $p \mid 3$, then the trefoil knot is p-colourable. The only prime number that divides 3 is 3, meaning that the trefoil is tricolourable, which we already knew to be true!

So, the system works.

An important thing to note is that the trefoil is only tricolourable. It is not, for instance, 5-colourable. This means that if we use the theorem, we can assign characteristics to a knot, like a checklist, to distinguish between different knots. Two knots could both be tricolourable, but maybe one is 7-colourable and the other is not. Therefore, they are not equivalent.

However, there is a glaring issue with the knot determinant. If two knots have the same determinant, that means they have the same colourability properties, so we need to generalise just one more time to reach our goal.

This knot invariant is the most powerful and useful one we have looked at so far. It is not a number or a set of colours, it is a polynomial, the Alexander polynomial.

To acquire this polynomial, the steps are similar to before, with one difference: orientation. Orientation is an arbitrary choice we can make on how we will travel around the knot, denoted by arrows. With this in mind we can define some new crossing notation in Figure 9.

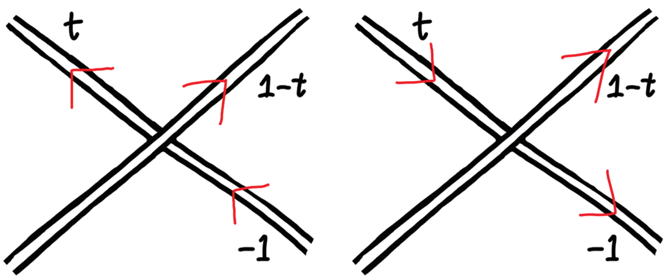

Figure 9: Right-hand crossing versus left-hand crossing.

Figure 9 shows the orientated crossing notation for our new matrix. Armed with this weapon, let us again use the trefoil as our target, and build this polynomial in Figure 10.

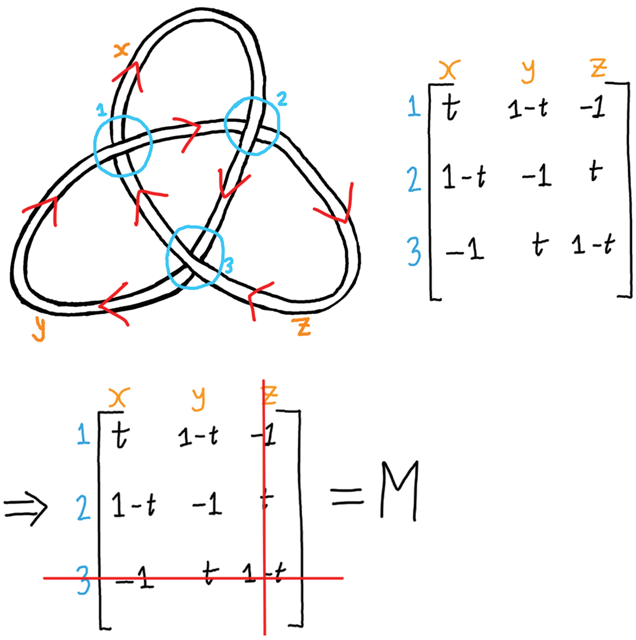

Figure 10: Finding the Alexander polynomial of a knot.

From Figure 10 we get the Alexander polynomial:

$$\det(\mathbf{M}) = -t^2 + t - 1.$$

Every knot has a unique Alexander polynomial, invariant up to $\pm t^m$. This way we have a more robust and generalised version of the knot determinant, which can fill in the gaps.

For our final result, we look back and tie up all the loose ends.

If you set $t = -1$ in the Alexander polynomial, you get the knot determinant. For the trefoil, the value is -3. This is divisible by 3 and hence the knot is tricolourable.

This works for each and every knot, making every result we have found a subset of the Alexander polynomial. From Reidemeister moves to our equivalence equation, we now have this Swiss-army polynomial to answer our original question.

And that is it. For most types of knots, we have a rigorous system to tell two of them apart. Mirror images of knots and complicated variations aside, we have finally achieved what the first knot theorists dreamt about. We have made Lord Kelvin proud.

It was a century before science finally caught up to our trailblazing knot theorists. They discovered that DNA is knotted in the ways that those pure mathematicians had been working on for years. Enzymes manipulate and knot DNA to edit and apply the formula of life. Not only can knots tell us about ourselves and genetics, they could write our future. Quantum physics has taken an interest in knots for some radical new theories, which could shape the world we live in.

Knot theory has advanced way beyond this article, with better polynomials, invariants and theorems being constantly created. However, it is still not known if there exists a knot polynomial that can easily distinguish all knots from each other, without *any* exceptions. This problem is an open one and I invite you to think on it.

All you need is a piece of string and a little imagination.

References

1 Adams, C. (2004) *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society.

2 Henrich, A. and Kauffman, L.H. (2014) Unknotting unknots, *Am. Math. Mon.*, vol. 121, no. 5, pp. 379–390.