Westward Ho! Musing on Mathematics and Mechanics

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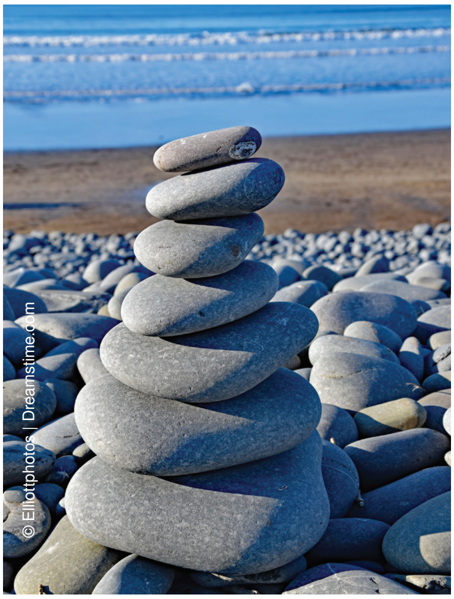
A chance meeting at a table tennis match, as recalled by Alan Champneys in his latest feature, led him down a path to find the connection between Bristol, decimal coinage and the mathematics of closed curves of constant width. The connection arises due to the unique shape of the 50p piece first introduced to the UK in 1969. The shape of the coin, and of the later-introduced 20p piece, is a so-called Reuleaux polygon. As well as being the world's first seven-sided coin, its curved sides are such that the coin is slot-machine compatible – it has the same width in any orientation. The man responsible for this design was Hugh Conway, then managing director of Bristol Siddeley Engines, the company that had built the engines of the supersonic airliner Concorde. The story leads us to encounter a square-hole drill bit, curious Chinese bicycles, a misnamed number game, Euler's hedgehogs, myths about manhole covers and the history of the Bugatti motor car.



Conway, Concorde and curiously curved coins

One of my main winter pursuits is playing table tennis in the Bath and the Bristol leagues. Over the years, I have not only relished the healthy competition but also the opportunity to explore various venues, such as village halls and social clubs, typically tucked away in remote spots. I also get to meet some delightful West Country characters.

On one such evening, towards the end of last year, I encountered, on the other side of the net, a collector of rare and interesting coins. He introduced himself and said that he had enjoyed the mathematical modelling exercise I had written on the geometry of the 50p piece [1].

Photo: Mr Brushwood

Figure 1: Rolling a 50p on a flat surface. The uppermost part of the coin remains a fixed height above the surface. Following [1].

It seems my opponent, who is known online as Mr Brushwood, was something of an expert on 50p coins, having written a rather unique manuscript chronicling the history of the coin [2]. As part of his hobby, he also adorns coins with coloured lacquer. So, the umpire's coin for the evening was a 50p coin whose reverse is a red, blue and green depiction of a table tennis rally (see Figure 1). Such coins (uncoloured) were originally issued as one of 29 different designs commemorating the London 2012 Olympic and Paralympic Games (including a football-related one that purports to explain the intricacies of the offside rule).

After our game (modesty forbids me from revealing who won), he asked if I knew the West Country angle to the history of the 50p piece. I did not. It seems the shape of the coin was designed by aerospace engineers in Filton, the area north of Bristol famed as the UK birthplace of Concorde and many other iconic aircraft. With a little bit more research, I discovered the story contains a good deal more interesting mathematics than I had realised.

First, let me describe the modelling exercise that Mr Brushwood was referring to. This free schoolteacher resource is one of a series [1] developed, in collaboration with the charity Mathematics in Education and Industry (MEI), to promote the kind of mathematical modelling that we teach on the engineering mathematics degree at Bristol. The problem gets the students to explore how, despite not being circular, the seven-sided 50p piece will roll perfectly on a flat surface. Each side is formed of a circular arc whose radius is that of the width R of the coin. So as it rolls, the coin alternates between pivoting on one of its corners – in which case the distance to the uppermost point is R – and rolling on one of its curved sides – in which case the opposite corner, a distance R away, forms the uppermost point. Thus, the outline of the coin is an example of a closed curve of constant width (Figures 1 and 2). The exercise then goes on to invite students to see that any odd-sided equilateral polygon with sides formed from circular arcs has this constant-width property. But not so for even-sided polygons.

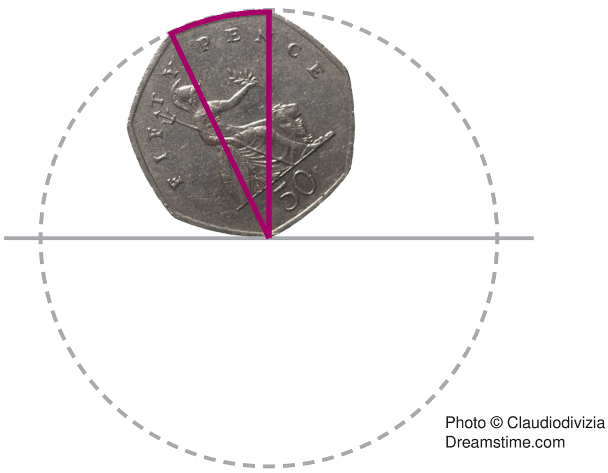


Figure 2: The circle described by the outline of the 50p piece pivoting on one of its corners. Following [1].

The IMA Maths Careers website also contains an interesting exercise [3] inspired by the 50p piece, proposing a measure for *roundedness* as the difference in radius between the maximum inscribed circle and the minimum circumscribed circle. See also Chapter 10 of [4], but more of that later.

But what of the history of the 50p piece and its West Country connection? By the early 1960s, the scientific and business communities were calling for a more rational replacement to the UK's antiquated coinage. There were 12 old pence to the shilling, 5 shillings to the crown and 4 crowns to the pound (oh, and 21 shillings to the guinea, an alternative to the pound used mostly for luxury goods). Coins of rather strange denominations had unusual names: the farthing (a quarter of a penny), ha'penny (half of a penny), thrupenny bit (three pence), tanner (six pence), florin (two shillings) and the half-crown (a tanner plus a florin).

Decimal currency was fully adopted in 1971, but the process of decimalisation began some ten years earlier when the Royal Mint invited artists to submit designs for the new decimal currency. The competition was won by Christopher Ironside, whose iconic design for the reverse of what was to become the new 50p piece is shown in Figures 2. The run-up to the revolutionary change was to be managed by a body called the Decimal Currency Board. It was quickly decided that the new denomination lower than the pound was going to be the new penny (plural pence), equal in value to 2.4 old pence, so that there were precisely 100 new pence to the pound.

The first new coins, for 5 and 10 new pence, were easily accepted as they were equal in value to shillings and florins, respectively (which remained legal tender until the early 1990s).

Figure 3: Mr Conway measuring a new 50p coin in 1968.

The introduction of the 50p coin presented more difficulty though. It was to be largest denomination coin ever issued for general circulation in the UK, replacing the old ten shilling note. For the smaller denomination coins, the Decimal Currency Board had decided to retain the useful property that bags of 'copper' or 'silver' coins could be evaluated simply by weighing them. So, the new half penny coin weighed half that of the new penny, which in turn was half that of the new 2p coin. Similarly, since a shilling weighed half a florin, so the new 5p would weigh half a 10p coin; these coins were identical in size and shape to their predecessors.

If this weight principle were to have been continued to the 50p piece though, each coin would have needed to weigh a massive 67.5 g, which would be like carrying a medallion in your pocket. To put this in perspective, you may recall that coins in the UK were successively reduced in size and weight in the 1990s, following the introduction of the 20p and £1 coins in the 1980s. By comparison, the current 50p piece weighs just 8 g.

Nevertheless, the new 50p coin had to look and feel significantly different from the smaller denominations. In 1971, it represented a lot of money, about £8 in today's money, and mistaking the new coin for something of lower denomination would be a costly mistake.

The conclusion was soon reached that a different shape of coin was required. A solution adopted in many other countries was to put a hole in the middle of the coin. One story has it that such an idea was quickly dismissed by the board, because the law requires the monarch's head to be present on all coins, and no one wanted to be responsible for trepanning Her Majesty.

Non-circular coins were thought to be a possible answer. But the onset of automation within commerce led to a problem; the new coin would need to roll under gravity and to fit in slot machines, no matter in which orientation it was presented.

The only member of the Decimal Currency Board with any engineering experience was Hugh Conway (Figure 3), who in 1967 was president of the Institution of Mechanical Engineers. His day job was as managing director of Bristol Siddeley Engines, a company in the process of being taken over by Rolls-Royce. Conway sent the problem of the new 50p piece to his company's design office, the team that had previously produced engines for Concorde.

In 2019, on the 50th anniversary of the first mintings of the 50p coin, the *Bristol Evening Post* [5] tracked down Colin Lewis, who had been a member of the original design team. They quoted him as saying:

> Mr Conway had found in a mathematical textbook a formula for a non-circular shape of constant breadth, but this turned out to be a wavy-edged form which would not roll and which could not be measured easily.
>
> This is basically an equilateral triangle with a small circle centred on each apex and with a larger circular arc centred on each apex but tangential to each of the two opposite small circles. Wherever it was measured, the breadth of this shape was one small radius plus one large radius.
>
> I made a cardboard mock-up to illustrate the proposal which was accepted by Hugh Conway, who chose seven sides as a compromise between too radical a shape, which might not be acceptable to the public, and having too many sides, which would make it visually difficult to differentiate from a circle.

The chosen shape is an example of a so-called seven-sided Reuleaux polygon. It seems that the triangular shape Colin Lewis is referring to is the original Reuleaux triangle, named after the German engineer Franz Reuleaux, who used such shapes to build mechanisms for translating between different types of motion. But on reading the above quote, I am left to ponder what shape might have been in Mr Conway's mathematical textbook?

I tried googling Conway's coin and I did not find what I expected. I found an article dedicated to *sylver* (sic) *coinage*, which is a game invented by the celebrated algebraist and recreational mathematician John Horton Conway, who sadly fell victim to the first wave of the COVID-19 virus. The game, as described in [6], involves two players successively deciding on the denomination of a new coin to be placed into circulation. A new coin can only be proposed if it represents a sum of money that cannot be represented using arbitrary combinations of the existing coinage. Only integer-valued coins are allowed. Each coin must be of larger denomination than 1 currency unit, but there is no upper bound. The winner is the last person to introduce a legal coin.

Conway named the sylver coinage game after the British mathematician James Joseph Sylvester (1814–1897) who, among many other achievements, is thought to have introduced the term *matrix* to the mathematical lexicon.

Sylvester had proved:

Theorem. *Given two positive integers m and n that are coprime (that is, with no factors in common), then the largest number that is not expressible as a sum of non-negative multiples of m and n is* $(m-1)(n-1)-1$.

It is reasonably straightforward to show that the sylver coinage game must always end, because after the second move, there can only be a finite number of possible coins still to be chosen. This is a trivial consequence of the above theorem in the case that the first two moves m and n are coprime, but can be argued more generally by considering the greatest common divisor of m and n.

Remarkably though, working out the optimal strategy in all cases remains an unsolved mathematical problem [7].

I digress. It seems that the concept of the Reuleaux triangle was already known long before its exploitation by the German engineer after whom it is named. The shape can be observed in certain Gothic windows, and it was used by Leonardo da Vinci to design a map projection method. Modern applications include a common shape for guitar picks (having three alternative pointed edges means they should wear out less quickly), bolts on US fire hydrants requiring special spanners and the cross-section of some pencils that are thought to be more ergonomic than traditional hexagonal ones (Figure 4).

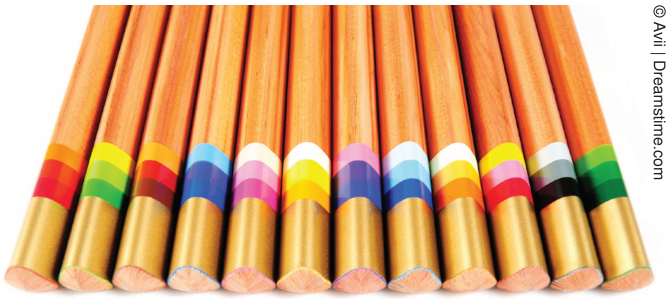


Figure 4: Pencils whose cross sections are Reuleaux triangles.

Another early application of the Reuleaux triangle is a patent by Harry Watts in 1914 for a bit that is capable of drilling a square hole (albeit with slightly rounded corners). The invention arises because a Reuleaux triangle is able to rotate continually while touching all four sides of a square hole whose side length is the width of the triangle (Figure 5). One quirk, as explained mathematically in [8] for example, is that although the Reuleaux triangle can roll perfectly, its centroid (aka the centre of mass) does not remain static as it does so. When rolling, the centroid actually transcribes a small circular-like shape (composed of four separate arcs of an ellipse). Thus, the centre of a Reuleaux drill bit rotates eccentrically, a feature that is incorporated within Watts's design.

This eccentricity of the centroid when rolling means that Reuleaux polygons are of little use as wheels suspended from a fixed axle. This did not deter Chinese inventor Guan Baihua, who in 2009 took the idea of a penny-farthing rather literally and built a rideable bicycle with a Reuleaux pentagon for its front wheel and a Reuleaux triangle for the rear. His ingenious design allows the axles to float using cantilevered forks.

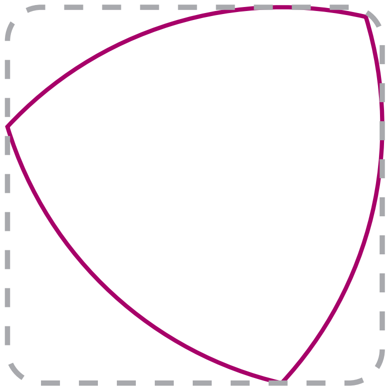


Figure 5: How to drill a 'square' hole.

The company Harry Watts founded with his brother in Wilmerding, Pennsylvania, still sells square-hole drill bits to the present day, as well as bits for drilling pentagonal, hexagonal and octagonal holes. Square, or rectangular, holes are mostly commonly seen in woodwork, where they form the female part of a mortise and tenon joint. For a tight fit, such holes are typically required to have right-angled corners, and are more commonly drilled using a so-called mortising drill that combines a drill bit within a square shaped hole punch.

Not all equilateral triangles with curved sides seen in nature and technology are true Reuleaux triangles, despite some popular myths. The computer scientist David Eppstein, who maintains the 11011110 blog, has a series on 'Things that are not Reuleaux triangles' [9]. His examples include manhole covers, the shape of the Kresge Auditorium at MIT and the optimal shape of the Wankel engine, the curved sides of the latter being technically *epitrochoidal envelopes* rather than circular arcs.

Returning to Colin Lewis's recollection of Mr Conway's supposed textbook with a formula for a 'wavy edged' curve of constant width, it seems that the circle and the Reuleaux solids are far from the only examples. In 1860, the French mathematician Joseph-Émile Barbier proved that any curve of constant width must have a perimeter equal to π times its width. It turns out there are many ways of constructing such curves. A good introduction can be found in the eponymous Chapter 10 of the beautifully eclectic book by John Bryant and Chris Sangwin FIMA, *How Round Is Your Circle?* [4, Ch. 10].

In what Martin Gardner [10, Ch. 18] called the *cross-lines method*, irregular polygonal curves of constant width can be constructed by taking a family of intersecting straight line segments in a plane, none of which are parallel. Ordering these lines by increasing slope, one then joins the ends of consecutive outermost intersections by an arc that is centred at the third point of intersection of the three lines involved. Repetition of this process, under certain 'reality' constraints, leads to a kind of irregular Reuleaux polygon (Figure 6). Perhaps, this was the construction Hugh Conway saw.

In fact, the mathematics of curves of constant width goes back to Leonhard Euler, who constructed a completely different family using so-called projective hedgehogs (I kid you not!). There are also known sets of curves of constant width that are constructed using elliptical arcs.

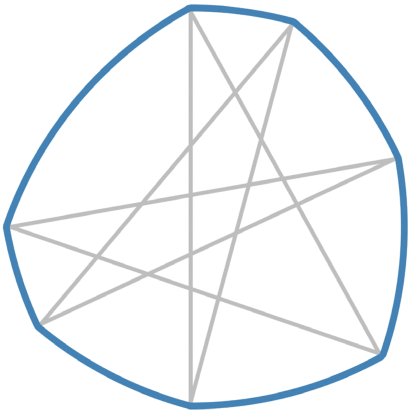


Figure 6: An example of an irregular Reuleaux octagon constructed using Gardner's cross-lines method. Credit: LEMeZza, CC BY-SA 3.0, `tinyurl.com/Wiki-Reuleaux`.

Most of these methods construct curves that are *piecewise smooth*, so that there are jumps in some derivative of the curve where each pair of pieces join. A natural conjecture might be that the circle is the only curve of constant width that is analytic; that is, whose derivatives exist to all orders at all points.

Not so. Several methods are actually known for constructing closed-form polynomials in two variables (x, y) whose zero set defines curves of constant width in the plane. The general question appears to remain an area of active research. For example, it was proved relatively recently [11] that the minimum total degree (highest value of $n = p + q$ of a term of the form $x^p y^q$) of such a polynomial is 8. Figure 7 plots one such example, which is the zero set of this polynomial:

$$\begin{aligned}(x^2+y^2)^4 - 45(x^2+y^2)^3 - 41283(x^2+y^2)^2 + 7950960x^2 \\ + 7950960y^2 + 16(x^2-3y^2)^3 + 48(x^2+y^2)(x^2-3y^2)^2 \\ + x(x^2-3y^2)(16(x^2+y^2)^2 - 5544x^2 - 5544y^2 + 266382) \\ - 373248000. \quad (1)\end{aligned}$$

I could go on. There are three-dimensional analogues, including Reuleaux tetrahedra and so-called Meissner bodies, in which there are many unsolved mathematical problems. But, perhaps I will leave any exploration to a future Westward Ho!

Let me return instead to Hugh Conway. His heralding of a new philosophy to coin design, while generally well received, was not universally accepted. Seemingly, a retired colonel, Essex

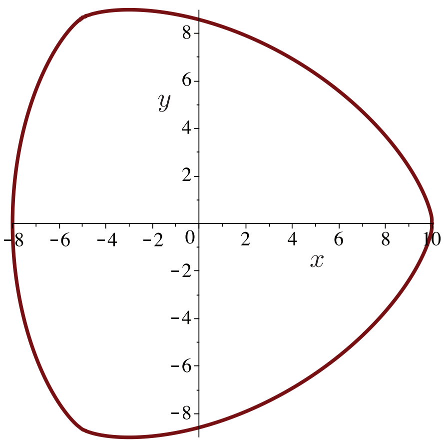


Figure 7: The zero level set of (1).

Moorcroft, formed a pressure group called the Anti-Heptagonists, reportedly referring to the coin as a 'heptagonal monstrosity' and 'an insult to our sovereign'.

Perhaps Conway's most enduring legacy is, in fact, to the history of motor sport. Upon retirement, he became arguably the world's leading authority on Bugatti sports cars. An avid collector and restorer of classic editions of the brand, he also amassed an extensive private collection of drawings, photographs and memorabilia relating to the life and work of the French-Italian motor pioneer Ettore Bugatti (1881–1947).

Hugh Conway died in 1989 just before he could fulfil a vision to open a museum to house his collection. Upon his death, the Bugatti Trust was set up by his son, also called Hugh, and the next year a museum was opened by HRH Duke of Edinburgh at Prescott near Cheltenham. The collection has grown and today provides a unique resource for studying the Bugatti story and the history of automotive engineering more generally. In 2018, Hugh Jnr was awarded an honorary degree by Coventry University in recognition of the unique relationship between the university and the Bugatti Trust.

Are there any conclusions to this tale? Is this merely recreational mathematics (not that there is any harm in that)? Perhaps not. In the era of additive manufacturing and computer-aided design, there seems to be a resurgence of interest in the mathematics of shape optimisation. I have recently encountered work considering such practical questions as the shape of a pair of counter-rotating stirrers for optimally mixing viscous fluids, the kinematics of a deployable synthetic heart valve and how to lay up a composite material to create an active prosthetic that will buckle into specific configurations.

So, perhaps, the next time you handle a 50p piece you should ponder, as the subtitle of Bryant and Sangwin's book suggests, on the enduring interplay between mathematics and engineering.

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