Historical Notes: The Pleasant Path of Curve Tracing

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Through the second half of the 19th century and into the 20th, courses in curve tracing populated the curriculum in British and American colleges and universities. In this context, curve tracing was limited to drawing curves [1, p. 1]:

> Whose equations are given in Cartesian coordinates [using only] the ordinary rules of Algebra as far as the Binomial theorem, the fundamental theorems of the Theory of Equations, and the general methods employed in Algebraical Geometry.

Significantly, no knowledge of calculus was required. Curve tracing defined in this way well preceded the 19th century, with later practitioners particularly singling out Gabriel Cramer's *Introduction à l'analyse des lignes courbes algébriques* from 1750 as a source for 'all later writers on the subject' [2, p. 43].

Evolving editions of George Salmon's *Higher Plane Curves* document a rising popularity. In the first edition from 1852, Salmon contended that [3, p. 128]:

> It may be proper to give some examples of the method of tracing the figure of a curve from its equation.

This was followed by a short section on the subject, referring to his *Conics* for the process of substituting numerical values of x into the equation. Two additional examples supplemented 'those which incidentally occur in the course of these pages' [3, p. 129]. After collaborating with Arthur Cayley for the 1873 edition, curve tracing became a much more prominent feature, carrying into the third edition in 1879. The number of examples tripled and the authors declared that [2, p. 43]:

> There is scarcely any exercise more instructive for a student than the tracing of curves, and more particularly those in which the equation contains one or more parameters which assume a succession of different values.

Cayley provided an even stronger incentive in his *Encyclopaedia Britannica* article on 'Curve', where he argued that [4, p. 717]:

> There is no exercise more profitable for a student than that of tracing a curve from its equation, or say rather that of tracing a considerable number of curves.

Cayley explained that the student should begin with 'purely numerical equations' where y is an explicit function of x, then progress to equations 'involving literal coefficients' and finally equations:

> Such that neither coordinate can be expressed as an explicit function of the other of them. [These would] require and serve as an exercise for the powers of an advanced algebraist.

With the ringing endorsement of the University of Cambridge's most famous mathematician, Percival Frost's 1872 textbook on *Curve Tracing* was well timed. Frost promised that his book relieved the mathematics student of 'the dull work involved in his preparation for climbing heights' [1, p. viii]. Instead, the student would travel [1, pp. viii-ix]:

> Along a very pleasant path, on which he may exercise in an agreeable way all his mathematical limbs, and, if he keeps his eyes open, may see a variety of things which it will be useful to have observed when his real work begins.

The metaphor recalled the well-trodden walks of Cambridge, where Frost worked, as well as the slopes and turns of the book's contents – curve tracing.[1] Whether the textbook lived up to its ambitious preface, it proved a publishing success. The fourth edition outlived the original author.

Frost highlighted the intrinsic value of curve tracing so that students would [1, p. vii]:

> Be strong all round … in all sorts of analytical processes and geometrical artifices.

In addition, the subject informed a wide variety of practical applications from 'Statics, Engineering, and Crystallography' to 'Optics and Astronomy' to 'the Lunar and Planetary Theories' [1, p. ix]. With attention to local needs, the textbook met the demand of examiners who had complained [1, p. vii]:

> Both of a want of power of work and of a want of individuality in the manner in which particular problems are attacked.

But curve tracing was not limited to Cambridge examinations. One of Cayley's students, Charlotte Angas Scott, imported the discipline to the United States. Between 1891 and when she retired in the mid-1920s, Scott designed and regularly offered a course on curve tracing at Bryn Mawr College for second-year mathematics students as well as any graduate students who did not yet have the requisite knowledge. In turn, her students expanded the tradition. At Mount Holyoke, Emilie Martin taught a geometry course [5, p. 65]:

> Dealing with properties of curves, curve tracing, and transformations.

In 1927, Mary Haseman typed up a short paper on curve tracing for the University of Illinois, where she worked as an instructor [6]. Haseman, who had completed her doctorate in knot theory at Bryn Mawr in 1918, credited Scott's lectures as informing her treatment of the subject.

In 1918, R.J.T. Bell, lecturer in mathematics at the University of Glasgow, observed that in the 45 years since Frost's first edition [7, p. xi]:

> Graphical work has taken an increasingly prominent place in mathematical instruction.

Yet, Haseman contended that few American institutions of higher education offered courses in curve tracing,[2] so students were unjustly [6, p. 1]:

> Required to trace only curves which can be traced with the meagerest knowledge of algebra, namely, by substituting values for the independent variable and determining values of the dependent one.

This laborious process always worked, but remained a last resort for the trained curve tracer.

Many of the strategies remain familiar: finding symmetries, examining asymptotic behaviour and drawing tangent lines. Others have become obscure, in particular Frost's analytical triangle and Scott's method of exclusion. The techniques are best animated by example. In accord with Frost's athletic metaphor, the following are selected as a gentle warm-up rather than to exhibit feats of strength.

Frost's analytical triangle

Frost's analytical triangle is based on the work of Jean Paul de Gua de Malves, which is in turn 'a modification of Newton's parallelogram' [7, p. 117]. It is useful:

> As a machine for saving the trouble of the comparison of the relative magnitude of the different terms of the equation of a curve, at an infinite distance, and in the neighbourhood of the origin, when the curve passes through it.

To construct the triangle, draw a right-angled isosceles triangle, 'divide the hypotenuse into as many equal parts as the degree of the equation' [7, p. 119] and draw parallel lines from these divisions to each of the two legs, $0x$ and $0y$ (Figure 1). Number the equal divisions along each leg from 1 (at the vertex 0) to n, then [7, p. 119]:

> Each point of intersection of such lines corresponds to a term of the complete general equation of the nth degree.

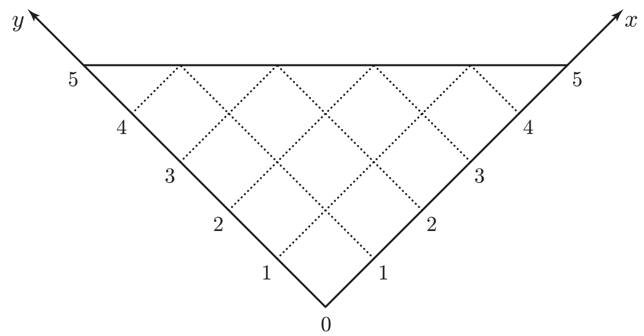


Figure 1: The analytical triangle, adapted from [7, pl. vi, fig. 21].[3]

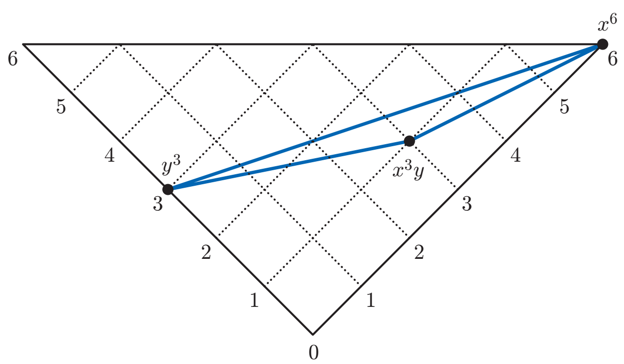


Figure 2: Frost's figure (adapted from [7, pl. vi, fig. 22]) of the intersections for (1) on the analytical triangle.[3]

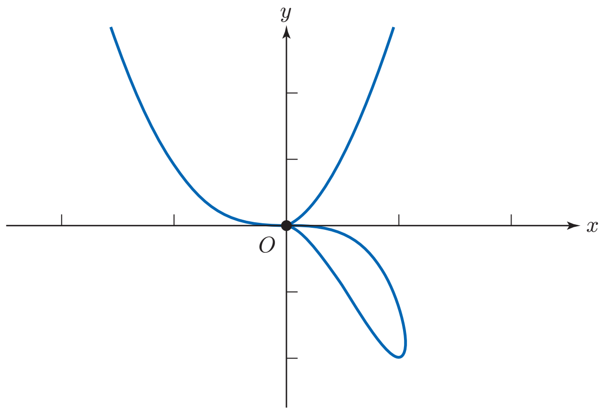


Figure 3: The curve (1), adapted from [7, pl. vi, fig. 23].[3]

An equation can then be placed upon the triangle by marking each intersection with a circle to designate a term of the equation. For instance, given the equation

$$x^6 + 2a^2x^3y - b^3y^3 = 0, \tag{1}$$

circles will be placed at the intersections corresponding to x^6, x^3y and y^3, respectively (Figure 2). These circles are then joined to form a convex polygon.

Among the valuable properties of the analytical triangle, if the straight line L, determined by any two circles, meets the sides of the analytical triangle $y0x$ then the remaining terms of the original equation will vanish compared with the terms forming L, either

(a) when x and y are infinitely great and the remaining circles lie on the same side of L as the right angle at 0, or

(b) when x and y are infinitely small, there are no constant terms in the equation and the remaining circles are on the opposite side of L compared to the right angle 0.

Frost demonstrated the 'truth' of these properties but dedicated most of the chapter to examples.

In the first example (1), consider the line corresponding to $x^6 + 2a^2x^3y = 0$. The remaining circle is on the opposite side of L compared to the right angle 0, so, from property (b), we expect this equation to determine behaviour through the origin. The equation can be reduced to $x^3 + 2a^2y = 0$, which Frost illustrates in the body of the text with a small cubic curve.

By the same argument, the line corresponding to $2a^2x^3y - b^3y^3 = 0$ can be interpreted to find the form of the curve near the origin. Frost simplifies this equation, writing $2a^2x^3 - b^3y^2 = 0$ and providing a small cusp alongside the text.

Finally, for the line corresponding to $x^6 - b^3y^3 = 0$, the remaining circle is on the same side as the right angle 0, so, by property (a), this side denotes the existence of 'the infinite branch,' in parabolic form, $x^2 - by = 0$. In the accompanying plate [7, pl. vi], Frost puts together these three pieces of information to trace the curve (Figure 3).

Frost proceeded into increasingly complicated examples, concluding [7, p. 131]:

> That with equations of high degrees, the use of the Analytical Triangle is almost indispensable.

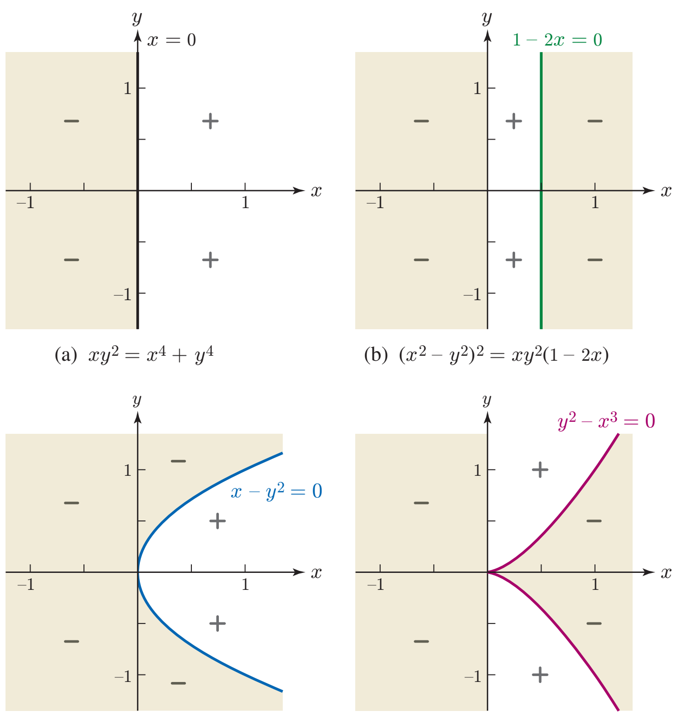
He then offered further cases for the reader to work out on their own.[4]

Scott's method of exclusion

Haseman noted that she had not found the method of exclusions 'anywhere in the literature' and believed it originated with Scott [6, p. 20]. The 'powerful tool in the tracing of a curve' rested on the observation that a curve [6, p. 17]:

> Represented by a function $F(x, y) = 0$, divides the plane into regions in which the $F(x, y)$ will be either positive or negative.

By factoring a given equation into different combinations of products, one could identify regions cut off by 'straight lines, conics, and the simplest of cubics' where the curve was excluded. Overlaying these excluded regions could eventually create a template onto which the curve could be placed.



(a) $xy^2 = x^4 + y^4$ (b) $(x^2 - y^2)^2 = xy^2(1 - 2x)$

(c) $x^4 = y^2\,(x - y^2)$ (d) $y^4 = x(y^2 - x^3)$

Figure 4: The method of exclusions, adapted from [6, figs. 6–9].[3]

The process is neatly captured in a series of figures illustrating

$$x^4 - xy^2 + y^4 = 0. \qquad (2)$$

Haseman rewrote the equation in four different forms. First in Figure 4(a) the equation $xy^2 = x^4 + y^4$ shows that x must always be positive, so the area $x < 0$ is excluded and shaded accordingly.

Rewriting the equation as $(x^2 - y^2)^2 = xy^2(1 - 2x)$ similarly shows that $x(1 - 2x)$ must always be positive. Since x is already positive, this means the area $1 - 2x < 0$ is excluded as shown in Haseman's second diagram in Figure 4(b).

When the given equation is in the form $x^4 = y^2(x - y^2)$, Haseman determined that $x - y^2 > 0$ and so excluded the left side of the parabola, drawn in Figure 4(c).

Finally, writing the equation as $y^4 = x(y^2 - x^3)$ combined with the earlier observation that x is positive shows that $y^2 - x^3 > 0$, which is illustrated by the shading in Haseman's Figure 4(d).

Haseman could now combine these results. The curve traced by the dashed line lay in the unshaded regions of Figure 5.

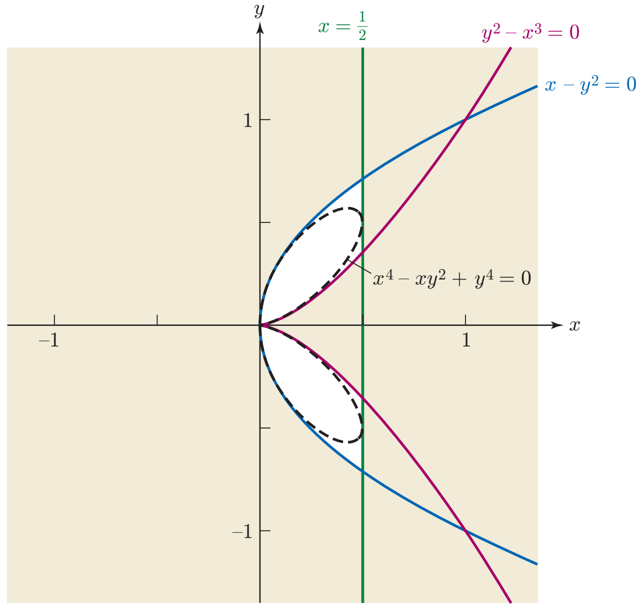


Figure 5: The curve (2), adapted from [6, fig. 10].[3]

While Frost and Haseman abjured 'ingenious' methods [6], there is something of the virtuoso in the selection and execution of examples in curve tracing. As Frost warned, the inverse problem (that of finding an equation for a curve of given form) [7, p. ix]:

> Will prepare the student for the disappointment which, having perhaps a wrong notion of what is meant by calling mathematics an exact science, he will feel in the conflict of theories by which it is attempted to reconcile the results of experiment in such subjects as Heat, Light, Electricity, and Molecular action generally.

The subject then, was admittedly, and even purposely, difficult. Still, Frost's book is very readable, containing 'detailed discussion and the diagrams of so many beautiful curves' [7, p. xi], and is freely available to today's reader in a digitised form.

Over the course of the 20th century the paper machines of analytical triangles and the method of exclusion were gradually replaced by graphing calculators and computer software.[5] Cheap and abundant digital technology challenges any practical value of plodding humans plotting points. Yet a focus on efficiency alone sidesteps 'the absolute necessity of developing skill and power' in mathematics provided by the tedious delight of curve tracing [7, p. vii].

Notes

1. On the central role of physical exercise, including vigorous outdoor walking, for students at the University of Cambridge, see Chapter 4, 'Exercising the Student Body: Mathematics, Manliness, and Athleticism' [8, pp. 176–226].
2. Among the few colleges offering such courses at this time were Bryn Mawr, the Massachusetts Institute of Technology and Vassar College.
3. Artist: Ron Weickart, `network-graphics.com`
4. A more contemporary treatment of Newton's polygons, including excerpts from Newton's correspondence with Leibniz and Oldenburg in which the technique was first developed, can be found in [9, pp. 370–385].

5. On the idea of 'paper tools' in the sciences, see [10]. For the development and adoption of graphing calculators in mathematics instruction in the United States, see [11, pp. 302–317].

References

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