Is Mathematical Education on the Right Track?

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"THE advantages of a classical education are two-fold—it enables us to look down with contempt on those who have not shared its advantages, and also fits us for places of emolument not only in this world, but in that which is to come." (Thos. Gaisford, 1779–1855, Good Friday sermon.)

"Their (practitioners of other subjects) criticisms of the teaching of mathematics may be accurate but nevertheless invalid." (SMP Pamphlet on Manipulative Skills in School Mathematics, 1974.)

The last couple of decades have witnessed an enormous spate of proposals for and trials of revolutionary mathematical curricula, counterpointed by weighty criticism that these are failing to disseminate numeracy (understanding of mathematical concepts and techniques, coupled with fluency in their sensitive and accurate use) more widely and deeply throughout the community. I am trying to see if suitable tests applied at some points where streams mingle can help to elucidate the virtues of different systems; but the field is now in such chaotic disarray, partly internally induced, partly because of general educational, social and political attitudes, and so bedevilled by the influence of strategically positioned pressure groups, that any feedback may take longer to obtain and correct for systematic distortion than is desirable in such a vital matter. In a civilisation that can only subsist in its present numbers and maintain the quality of its life by the application of scientific methods to its production and social processes, the fundamental importance of mathematics[1] makes it imperative to reinforce such investigations by other means, lest too much damage be done before any errors can be fully identified and measured by traditional empirical methods. So this paper examines three aspects of mathematical education: Why?; What?; How?; to see if existing evidence and arguments justify present trends. It would be disastrous if mathematicians, having taken over the task of providing the *lingua franca*, and the basic ideas and analytical techniques of science from the classicists, emulated the arrogance that isolated their predecessors, and so destroyed the effectiveness of the service they could and should provide.

Why teach mathematics?

The roots of much current confusion lie here, and illustrate Eddington's constant warning that the most likely place in which to find mistakes in any scientific thesis is in assumptions made before the argument begins, which are unstated, or unchallenged, because they are too familiar or fashionable.

No subject should, as a compulsory subject during the period of compulsory schooling, be taught as a "subject in its own right." The only moral justification for the teacher generation imposing particular studies on their pupils is that they are essential components in an intelligently designed, integrated kit of ideas, knowledge and techniques that the pupils will need for coping with adult responsibilities, both for their continuous education (formal or informal) throughout their lives, and for fitting themselves, even as rebels, into a survival pattern suitable for a species that claims classification as *homo sapiens*.

Similarly, no art can expect to be subsidised to the extent necessary for scanning the population thoroughly for high ability, and for generating sufficient general competence to enable its achievements to penetrate and fertilise communal thinking, if it offers no higher return than the aesthetic pleasures of, say, ballet or chess. Lest the magnitude of the task be forgotten, note that comparing typical timetables with total expenditure on education indicates that in the UK the cost of teaching mathematics and mathematically based subjects (fields in which many mathematicians are pressing for expensive improvements in quality and coverage) is already approaching the order of a billion pounds (£10^9) a year (projection from Annual Abstract of Statistics, corrected for current accelerated inflation).

Fortunately both conditions can be satisfied; for while no subject has any vested axiomatic right to its present standing, mathematics is fully entitled to its major place in the school curriculum as the main, basic language of science. So any entrant to a scientifically powered society must understand and be able to converse in it, if he is to follow many of the most important themes being discussed, make the most of any opportunities that come his way, and contribute to the fullest extent to the welfare of the community in his discharge of the duties incumbent on adult citizenship. It is as the only or most convenient medium for encoding and analysing many vital problems that the art can command both the attention of the individual student and the sponsorship of society.

Moreover, making its value as a means of communication its primary justification does not inhibit, it rather promotes other objects for which the teaching of mathematics is advocated; these follow as immediate corollaries, further developments, or symbiotic partners dependent on it. However, the converse is not true; so essential elements are in danger of being lost if the correct, natural order of priority is disregarded.

Mental growth

Many educationalists treat this as a separable object, distinguishing sharply between the development of concepts and the development of symbolism for handling these concepts conveniently. Even in the simplest operations, the abstract splitting apart of two facets of

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one integral problem is practically impossible; in more advanced spheres it can become a dangerous will-o'-the-wisp, as mental constructions become quite unmanageable without adequate language to characterise them, so they grind to a halt. This is demonstrated by the invariability with which leading thinkers have to adopt new (at first almost incomprehensible) jargons, or modify existing conventions even at the risk of causing confusion. Reaction against the rote learning of uncomprehended verbiage must not be allowed to obscure the fact that steadily deeper consideration of the significance of words or other forms of representation is still the best approach to understanding and refining the concepts designated by them, and, indeed, the processes of thought themselves.

Technical services

The ability of mathematics to provide the most powerful analytical and computational services to commerce, industry and governments depends in the first place on the development of skill in articulating their problems, and in translating them between different phases. Usually the most difficult task in any applied field is turning the patron's often vague aspiration into technical terms accurately, so that its feasibility can be determined, and any proffered solutions assessed with some degree of precision. In such work, clear understanding of the significance and limits of formulae and processes is essential; more errors arise from selecting the wrong approach or transcribing it carelessly than in subsequent operations. These errors can be extremely difficult to detect and serious in their consequences in the multi-disciplinary projects that most social affairs involve.

Finding and training mathematicians

This function is often advanced as dictating the need for drastic changes; yet basing the school curriculum on utilitarian rather than sectarian lines offers better chances of success. Those predestined by genetic make-up or social pressures will become mathematicians regardless of approach, provided they are not bored; any course intelligently taught will stimulate their natural bent. If there is a problem in attracting an adequate flow of professional mathematicians to maintain the quality of teaching and research, it lies in convincing a higher proportion of the generally gifted that mathematics is the sector of their interests that they should pursue as a full time career. Such students are more likely to respond to courses that emphasise the social value of the subject as a means of communication with the widest range of applicability, than to an over-arrogant assertion of the virtues of its own internal interests.

Intellectual delight

Even the appreciation of mathematics as a subject in its own right depends at every stage on growing fluency in the mastery of its vocabulary and grammar; without this, too much time is lost in looking up meanings and checking constructions to leave any for savouring its quality.

The acceptance of its communication rôle as the primary object of teaching mathematics in schools leads to some awkward corollaries when discussing what mathematics should be included at each stage, and how it is best taught; but as the reforms that mathematicians are trying to achieve cannot be attained without strong backing by the general public, they would be well advised to recognise its legitimate interests, rather than promote their own specialist interests by such dubious tactics as the Sputnik scare.

What mathematics should be taught?

Mathematics must become a universal, native language of society if the community as a whole is to understand and control the scientific system it has created, and on which it now depends for many essential services.[2] Otherwise *hoi polloi* can only yield grudging deference to a new priestly caste speaking a language it cannot understand, and hence can easily be persuaded to suspect. But if this consummation is to be reached, school mathematics, particularly during the general phase of education, must satisfy the following three conditions.

1. The whole must be designed as a branching system from a common core. Unless this is done, specialists in remote sectors will lack rapport with other groups, and hence the support they need to pursue their own studies effectively, by feeding their results back into situations where they can prove their worth, and so attract continuing sponsorship.

2. Below each branching point, the choice of items to be included must be determined by the (highest common factor of the) needs of all those involved, not dictated by specialist interests at higher levels that may wield disproportionate influence on the administration of the system.

3. At every stage, priority must be given to acquiring fluency in the utilisation of those aspects covered. This is the primary consideration for the time and energy spent in learning any language; so sufficient exercises must be included in any course to convert the scholar's initial assent to new ideas into the familiarity that facilitates their use freely, accurately and generally.

There is no evidence of any growth in human mental capacity over the last few generations; indirect pointers from comparative genetics suggest that any change is probably negligible over such a period in such an unselective population as the human race. So any changes in curricula must eliminate items from earlier ones equivalent in content to any new ones introduced, if this is not to entail a reduction in the effort available for students to acquire personal proficiency. Indeed, where educational systems have extended their coverage and become less selective, the content of the common core may have to be reduced.

Loss of much of the substance of traditional syllabuses would be retrograde, as earlier reforms had left little that was not practically useful. Some room for additions can be gained by pruning where simplification of the route by which results are obtained is possible, for example establishing Euclidean spatial relationships by non-axiomatic methods, or reducing the span of work on units where rationalisation has taken place. Yet it seems doubtful whether the scope these provide is enough to accommodate all the new topics now proposed, some of general concern, but many only to professional mathematicians, scientists or technologists. Many students exposed to an excessive mass of ideas are unable to digest them, and fail to learn how to apply them themselves; yet it does not seem difficult to devise a more serviceable common core that could obviate this without inhibiting the penetration of the higher mathematical fliers more deeply and rapidly, provided rational grouping of scholars within any band or institution is arranged. One possible way is illustrated in Appendix I.

How should mathematics be taught?

Obviously, if some revolutionary system of presenting mathematical concepts and techniques to students were discovered that enabled them to be assimilated more readily, changes in the curriculum to "cash in" on such a "god-send" would be inevitable. As such a revelation is claimed by some promoters of new syllabuses, some of the advantages pressed and inconveniences ignored must be examined to assess their merit.

Teaching structures

There is nothing novel about teaching structures; it was once the standard approach to languages, particularly the dead ones, which by their condition are most susceptible to dissection and analysis. Its success with small groups of selected pupils showed that it can work well in certain circumstances, and what these are; so there is no reason to doubt that, in an educational system that segregated its mathematical students carefully, the top echelons might profit by this method. Yet its defectiveness as a general method of learning languages was manifest, and aggravated the difficulty of arranging selection rationally and equitably, without tension between or isolation of different streams. With all but the most gifted children, the absence of background knowledge on which to hang and exercise the abstract structures made lessons degenerate into rote memorisation that contributed little to developing understanding or fluency of use; so the majority remained uninspired, and refused to maintain contact with the subjects, to extend and utilise their knowledge once free of the classroom. Such a reaction does not conduce to the deeper penetration of any art.

It is relevant to note that this traditional English response to foreign languages is now being overcome in that field by abandoning the structural approach, and basing teaching on more practical methods, such as the language laboratory, and more utilitarian syllabuses.

Natural sequences

Although now carried on by more sophisticated means, this also is a continuation of a long tradition, of the school of teachers that has tried to match material and programmes to the child's natural, inherent characteristics at each stage, as against that which stressed the virtue of forcing down unpalatable medicines. Yet one must wonder whether the very power of the machinery now often deployed may not destroy the conditions necessary for obtaining valid information, the massive adult presence creating biases towards or highlighting responses stemming more from the observers' interests than from the natural reactions of children in freer situations, and learning more from relations with their peers. Moreover, is it natural or reasonable to concentrate attention on the isolated individual's pattern of mental development when investigating a social phenomenon, such as language, in a highly organised and socialised species, such as man? The most cogent evidence for the natural order of mathematical development must be the order in which mathematics developed in fact in nature, which is reflected in traditional curricula.

Social change

Two critical aspects of extending mathematical education deserve deeper consideration, but are often forgotten because, as in mountaineering, the heights outstanding individuals can scale are so dazzling that they blind observers to the vast pyramids of earlier and lesser mortals on which their ascents are based. These are as follows.

1. The degree of disorder tolerable within any community. Some freedom to experiment with a teacher's own ideas and pursue his personal interests is essential for maintaining enthusiasm, and the ferment from which useful reforms evolve; but this must be contained within reasonable limits if the majority of pupils are not to become confused about the purpose of their studies, lose heart and drop out. A considerable level of coherence throughout any system is particularly necessary in the mobile social flux demanded by an advanced economy.

2. The maximum rate of change that any system can stand without internal stresses becoming disruptive. Most only remain stable within limits, becoming stagnant if the rate of adaptation is too low, breaking up if too high. Educationalists need to heed Huxley's warning[3] that in the present phase of evolution the methods of controlling the release of change will have to be consciously planned and executed if the balance between stability and change necessary for survival is to be struck.

In "New Mathematics" two very differently motivated reform movements have become confused. The first stems from a pupil oriented concern to purge the classroom of dull, indigestible fodder, and replace it by more tempting offerings that will encourage children to forage for themselves. As long as the need for sufficient practice, which can never be as thrilling as meeting new ideas and situations, is not forgotten, this is wholly admirable; it is applicable to any style of curriculum, but most readily to the most realistic. The second owes its main drive to teachers of mathematics wanting to introduce topics that are professionally interesting to them, regardless of the utility of these in the great majority of situations where non-mathematicians need some mastery of mathematical language if they are to discharge their rôles intelligently. It seems questionable whether attempts to order the general, ordinary level of education mathematical syllabus on these lines is morally justifiable, or in practice compatible with the deeper penetration envisaged by Sir James Lighthill, or the broader understanding urgently and rightly demanded by Sir Frederick Dainton.

References

1. Dainton Committee Report, "Enquiry into the Flow of Candidates in Science and Technology into Higher Education," Council for Scientific Policy, H.M. Stationery Office, Cmnd. 3541, 1968.
2. The logical end-point of Sir James Lighthill's 'deeper penetration,' Presidential Address, Second International Conference on Mathematical Education, Exeter, 1972.
3. Huxley, J., "Toynbee and time-scales," *in* "Essays of a Humanist," Pelican, 1966.

Appendix I

Rationalisation of GCE O-level mathematics

Mathematics (Common Core)

Numerical relationships:

Weights and measures, including conversion of units, non-decimal scales
Fractions, ratios, decimals, percentages
Types of numbers (prime, composite, rational). Standard form
Approximations, limits of accuracy

Spatial relationships:

Length, area, volume—mensuration
(Non-axiomatic) Euclidean space, triangles (including Pythagoras), circles, polygons
Elementary solids (including the Earth as sphere)
Basic trigonometrical ratios, problems solvable with these
Simple loci
Scale drawing, conversions (including mapping)
Elementary Cartesian geometry
Simple vectors

Structure:

Elementary algebra, as generalised arithmetic
Formulae, manipulation, evaluation
Graphical representation, use and limits
Proportional sets, logarithms
Elementary linear equations (2 variables)
Elementary calculus, from graphical base
Introduction to set nomenclature, relationships

Social relationships:

Elementary probability
Elementary statistics—
  Graphical techniques
  Basic measures, mean, mode, median, quartiles
  Significance of different distributions

Mechanical relationships:

Elementary kinematics, relative motion
Newton's laws, friction
Elementary electricity, Ohm's law, power

Manipulative skills:

Accuracy, speed in computation; significance of sizes
Sketching, elementary geometrical constructions
Use of logarithms (4 figure), slide-rule, simple (unprogrammed) calculators

Additional mathematics (for those needing/taking A-level)

Sets, Venn diagrams, simple problems
Elementary logic, truth tables
Symbols as general representatives, idea of groups

Functional relationships, mapping, inverses, combinations
Differentiation, rates of change
Integration, areas, volumes
Differentiation and integration of simple functions

Algebraic operations on polynomials, higher degrees
Quadratic and 3 variable equations
Complex numbers
Matrices, elementary operations
Elementary series

Geometry of more complex curves (conics) and solids, comparative use of projective, Cartesian, polar methods
Affine transformations
Networks, topological relationships
Vectors—products, components

Iterative procedures
Further statistics, methods of correlation
Particle dynamics (elementary)