Abel Answers the Question of the Quintic

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This year marks the 200th anniversary of the solution of one of the most famous problems in mathematics: Abel's proof of the algebraic unsolvability of the general quintic equation. At the time of the proof's first publication in 1824, this problem had been open for over 250 years, so its solution by an unknown Norwegian mathematician in his early twenties was a major achievement. Yet despite this, few people today know much about Abel and even fewer have seen his ingenious proof. This article is, therefore, intended as a gentle introduction to both.

But first, a little background ...

Algebraic equations before Abel

Since antiquity, we have known the general quadratic equation

$$ax^2 + bx + c = 0$$

could be solved using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

although today's modern notation did not emerge until the 17th century. But it was not until 1545 that the Italian mathematician Girolamo Cardano (1501–1576) published the first universal method for solving the general cubic equation

$$ax^3 + bx^2 + cx + d = 0.$$

By use of the clever substitution $x = y - b/3a$, he simplified this equation to the form

$$y^3 + py + q = 0,$$

for which the solutions were found using the new cubic formula:

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This discovery prompted a further breakthrough by one of Cardano's students, Lodovico Ferrari (1522–1565): the creation of an elaborate method for solving the quartic equation.

By the mid-16th century, increasingly sophisticated formulae had been discovered that would give the solutions to any algebraic equation of degree $n \leq 4$ in terms of expressions involving the coefficients of the original equation, combined using only the operations of addition, subtraction, multiplication, division and the extraction of roots. Such techniques became known as solving an equation 'by radicals'. The obvious task now was to find a formula for the general quintic.

But for well over 200 years, the goal of finding a general algebraic solution to the quintic equation remained elusive. In a famous paper of 1770, Joseph-Louis Lagrange (1736–1813) proved that none of the techniques for solving quadratic, cubic and quartic equations would work when applied to the quintic [1, pp. 295–298], prompting the suspicion that a general quintic formula might not actually exist.

The first attempt to confirm this came in 1799, when the Italian mathematician Paolo Ruffini (1765–1822) published a book-length proof that the general quintic equation could not be solved by radicals. This proof was exceptionally long and complicated, and failed to convince the wider mathematical community. It is now agreed that, although somewhat convoluted, Ruffini's argument was basically correct but that it contained significant gaps [2].

In his proof, Ruffini employed an important idea from Lagrange's 1770 paper, namely that the algebraic solvability of equations was strongly connected to permutations of their roots. This inspired the famous French analyst Augustin-Louis Cauchy (1789–1857) to investigate permutations in greater detail.

In a landmark paper of 1815, Cauchy proved a theorem that would soon turn out to be of major importance [3, p. 9]:

> If a rational function of n objects takes fewer values than the largest prime less than n when the n quantities are permuted, it can take at most two different values.

It was here that Abel entered the story.

Figure 1: Ruffini (left), Cauchy (centre) and Lagrange (right) provided the foundations for Abel's proof

Figure 2: Abel's teacher Holmboe (left), his publisher Crelle (centre) and Abel (right)

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Abel: a more than able mathematician

Niels Henrik Abel was born into a Norwegian family of modest means on 5 August 1802. The second of six children of Søren Abel, a church minister, and his wife Anne, the young Niels received his earliest formal education from his enlightened and university-educated father. At the age of 13, he was sent away to the Cathedral School in Christiania (renamed Oslo in 1925) but did not thrive. However, with the arrival of a new mathematics teacher, Bernt Holmboe (1795–1850), in 1817, the young Abel soon began to excel at mathematics, although his performance in other subjects was not exceptional.

The death of Abel's father in 1820 led to mounting financial pressures, but with Holmboe's help and encouragement, Abel's mathematical prowess increased dramatically and he managed to obtain a vital scholarship. In 1821, at the age of 18, he entered Christiania's Royal Frederick University.

Abel had already turned his attention to the problem of the quintic equation, although he originally approached it from the opposite direction. Believing that he had come up with a general solution, he wrote to the then-famous Danish mathematician Ferdinand Degen (1766–1825). But when Degen pressed him for an actual example, Abel realised that he was mistaken, so he turned instead to trying to prove that such a solution was impossible.

In 1824, Abel published his first proof that the general quintic equation was unsolvable by radicals [4]. It was printed as a small booklet at his own expense, with the intention of using it in the manner of a business card to attract the attention of the great mathematicians of Europe, whose ranks he earnestly wished to join. The expense of printing was considerable, however, so to economise, Abel condensed his argument down to the main points, not all of which were proved in the space available.

Awarded a travel grant to further his studies, Abel left Norway in 1825 and, while in Berlin, met the German engineer and amateur mathematician August Leopold Crelle (1780–1855), who became a great champion of his work. In 1826, Crelle published a greatly expanded version of Abel's paper in the first volume of his *Journal für die reine und angewandte Mathematik*, or *Crelle's Journal* as it became widely known. This paper contained the full proof, with the details explained and proved at length [5]. Although there remained a few minor gaps, all of which were later filled, it was this proof that was eventually accepted by the mathematical community at large. Here then is a summary of Abel's main argument. (For an excellent, and far more detailed, analysis of Abel's proof, see [6].)

Abel's proof

He began by assuming that the general quintic equation *is* solvable by radicals (see Figure 3). Taking a general nth-degree equation:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0, \quad (1)$$

where n is a prime number, so that it cannot be factored, he proved that all algebraic solutions to it can be expressed in the form:

$$x = p + p_1 R^{1/n} + p_2 R^{2/n} + \cdots + p_{n-1} R^{(n-1)/n}, \quad (2)$$

where p, p_1, p_2, …, p_{n-1} are finite sums of radicals and polynomials, and the nth roots of R are irrational functions of the coefficients of the original equation.

Now, substituting this version of x into the original equation (1) and reducing the resulting expression to the form in (2), he obtained:

$$P = q + q_1 R^{1/n} + q_2 R^{2/n} + \cdots + q_{n-1} R^{(n-1)/n}, \quad (3)$$

which, since the expression in (1) was set to zero, will also equal zero. Abel then used a very clever argument to prove that, for this to be true, all of the q's must equal zero as well. This then enabled him to deduce that any solution x of the form (2) can be expressed purely in terms of rational functions of the roots. This fact had merely been assumed by Ruffini, resulting in one of the chief flaws of the earlier proof.

For a quintic equation, according to (2), its solutions should be of the form:

$$x = p + p_1 R^{1/5} + p_2 R^{2/5} + p_3 R^{3/5} + p_4 R^{4/5}, \quad (4)$$

in which, by virtue of what Abel had now proved, all of the component parts – p, p_1, p_2, p_3, p_4 and the various integer powers of $R^{1/5}$ – would be rational functions of the roots of the quintic equation.

Now recalling Cauchy's theorem on permutations from 1815, Abel considered the special case when $n = 5$. This gave him the following result [5, p. 77]:

> If a rational function of five quantities takes fewer than five values when the five quantities are permuted, it can either take two different values or one value.

He then turned his attention to the hypothetical solution (4) of the general quintic and considered what would happen if the five roots were permuted.

By clever use of Cauchy's theorem, Abel was able to show that each case led to an equation in which one side has 120 (or 5!) possible values, while the other side has only five or ten. Thus, no matter which possibility he chose, a contradiction was unavoidable. This meant that the supposed solution of the general quintic equation could not in fact exist, leading to the inevitable conclusion [5, p. 84]:

> It is impossible to solve the general equation of the fifth degree by radicals.

As a final remark, Abel closed his proof by noting an instant corollary:

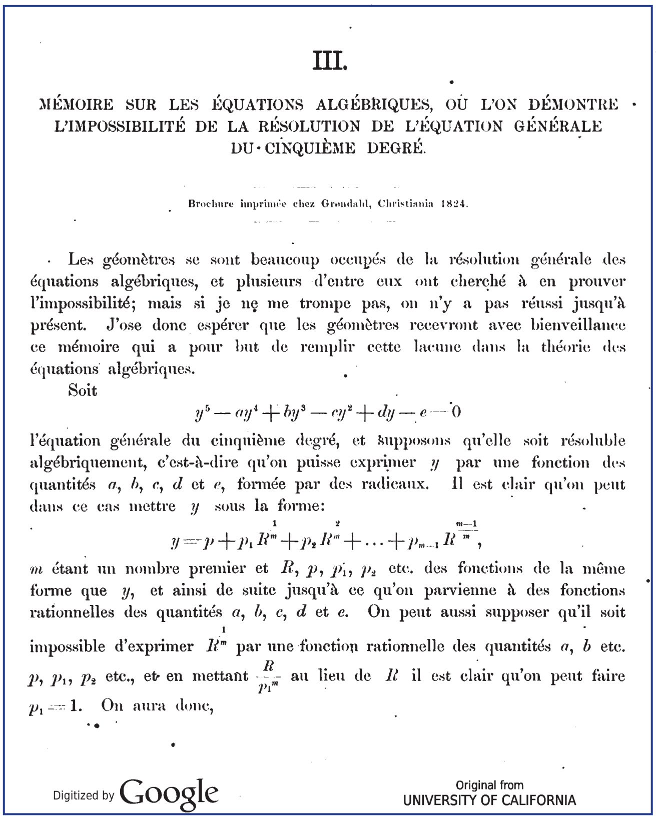
> It follows immediately from this theorem that it is similarly impossible to solve general equations of higher degree than the fifth by radicals.

Aftermath

In 1827, after 18 months of European travel, including extensive stays in Paris and Berlin, Abel returned to Norway in debt. Despite his resolution of one of the most outstanding problems of the time and despite having published several innovative papers, particularly in *Crelle's Journal*, including groundbreaking research on the theory of elliptic functions, the mathematical world still seemed largely indifferent to Abel's work. On his travels, he had failed to meet influential mathematicians such as Gauss and Legendre, and he had been unsuccessful in obtaining a paid academic position at home or abroad. Worst of all, while in Paris he had contracted tuberculosis, which in the absence of antibiotics was essentially an extended death sentence.

While friends such as Crelle promised to lobby on his behalf for a permanent job, Abel was reduced to part-time tutoring and taking out loans. As he continued to research and publish throughout 1828, both his health and finances continued to deteriorate. Then, on 8 April 1829, Crelle wrote with the good news that he had finally managed to obtain a position for him in Berlin. But it was too late. Abel had died two days earlier. He was 26 [7].

It was only posthumously that Abel's pioneering work was finally recognised. His innovative ideas were built on and extended by mathematicians such as Galois and Jacobi, and before long his name had been immortalised across mathematics via Abel's summation formula, Abel's limit theorem for power series, abelian integrals, abelian varieties, abelian groups and the like. In fact, Abel's name became so ubiquitous in mathematics that now, when used as an adjective, the word 'abelian' is no longer capitalised. The bicentenary of his birth in 2002 saw the inauguration of the Abel Prize; awarded to one or more outstanding mathematicians every year, it quickly became one of the world's highest mathematical honours. For details about this year's winner please see page 65.



III.

MÉMOIRE SUR LES ÉQUATIONS ALGÉBRIQUES, OÙ L'ON DÉMONTRE L'IMPOSSIBILITÉ DE LA RÉSOLUTION DE L'ÉQUATION GÉNÉRALE DU CINQUIÈME DEGRÉ.

Brochure imprimée chez Grondahl, Christiania 1824.

Les géomètres se sont beaucoup occupés de la résolution générale des équations algébriques, et plusieurs d'entre eux ont cherché à en prouver l'impossibilité; mais si je ne me trompe pas, on n'y a pas réussi jusqu'à présent. J'ose donc espérer que les géomètres recevront avec bienveillance ce mémoire qui a pour but de remplir cette lacune dans la théorie des équations algébriques.

Soit

$$y^5 - ay^4 + by^3 - cy^2 + dy - e = 0$$

l'équation générale du cinquième degré, et supposons qu'elle soit résoluble algébriquement, c'est-à-dire qu'on puisse exprimer y par une fonction des quantités a, b, c, d et e, formée par des radicaux. Il est clair qu'on peut dans ce cas mettre y sous la forme:

$$y = p + p_1 R^{\frac{1}{m}} + p_2 R^{\frac{2}{m}} + \ldots + p_{m-1} R^{\frac{m-1}{m}},$$

m étant un nombre premier et R, p, p_1, p_2 etc. des fonctions de la même forme que y, et ainsi de suite jusqu'à ce qu'on parvienne à des fonctions rationnelles des quantités a, b, c, d et e. On peut aussi supposer qu'il soit impossible d'exprimer $R^{\frac{1}{m}}$ par une fonction rationnelle des quantités a, b etc. p, p_1, p_2 etc., et en mettant $\frac{R}{p_1^m}$ au lieu de R il est clair qu'on peut faire $p_1 = 1$. On aura donc,

Figure 3: The opening of Abel's 1824 proof [8, p. 28]

Abel's mathematical career lasted for less than seven years, but the profundity of the ideas he produced continues to influence mathematical developments to this day. As the French mathematician Charles Hermite (1822–1901) wrote:

> Abel has left mathematicians enough to keep them busy for 500 years.

And it all began 200 years ago with Abel's answer to the question of the quintic.

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