Invariance Properties of Regular Polygons

Sean McKee FRSE CMath FIMA, University of Strathclyde

In this article elementary proofs of invariance results for equilateral triangles and regular polygons are provided. A generalisation of Carnot's perpendicularity theorem is given.

In an equilateral triangle, the sum of the distances from an arbitrary point within the triangle to the three sides, multiplied by the length of a side, is always constant and equal to $\sqrt{3}/2$.

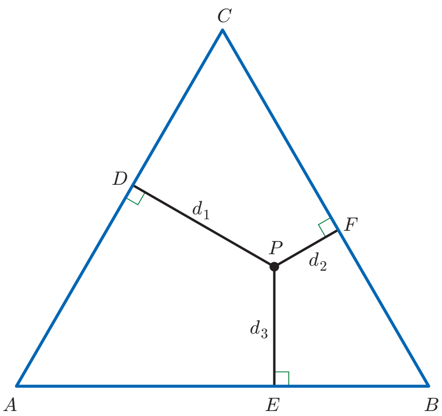

Figure 1: An arbitrary point P within an equilateral triangle with perpendiculars to each side.[1]

The proof of this result is surprisingly simple and relies on the formula for the area of a triangle. Consider Figure 1, which displays an equilateral triangle with an arbitrary point P within the triangle. Note that perpendiculars have been drawn from P to each side. Without loss of generality, we can assume that all the sides have unit length: it need only be demonstrated for similar triangles.

In Figure 2 we have included the lines AP, BP and CP. Now observe that the areas of the triangles APB, BPC and CPA, when added together, are simply the area of the triangle ABC:

$$\frac{1}{2}d_1 + \frac{1}{2}d_2 + \frac{1}{2}d_3 = \frac{1}{2}h = \frac{\sqrt{3}}{4}, \tag{1}$$

and the result is proved upon cancelling through by $1/2$.

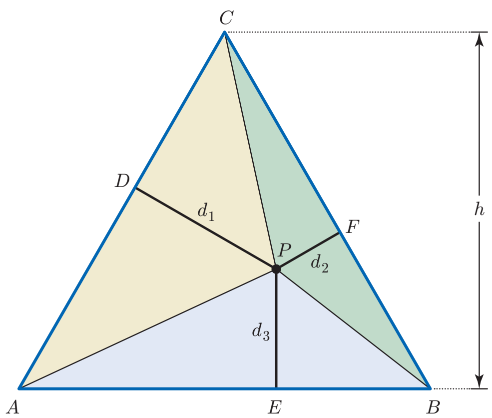

Figure 2: The same equilateral triangle with additional lines drawn from P to each vertex.[1]

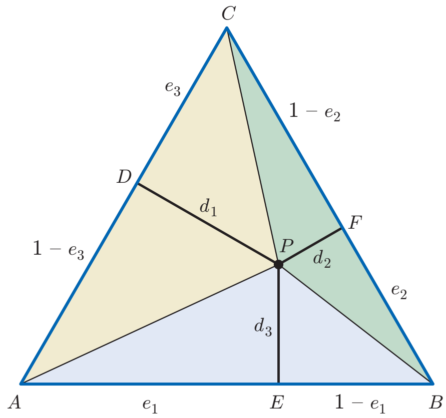

Figure 3: Equilateral triangle with side length prescribed.[1]

This result is known as Viviani's theorem. Vincenzo Viviani (1622–1703) was a pupil of both Galileo and Torricelli. It is, however, not the only invariance property associated with this triangle. Consider Figure 3 with $e_1 = |AE|$, $e_2 = |BF|$ and $e_3 = |CD|$. Repeated application of Pythagoras' theorem results in

$$|AP|^2 = (1-e_3)^2 + d_1^2, \tag{2}$$
$$|AP|^2 = e_1^2 + d_3^2, \tag{3}$$
$$|BP|^2 = (1-e_1)^2 + d_3^2, \tag{4}$$
$$|BP|^2 = e_2^2 + d_2^2, \tag{5}$$
$$|CP|^2 = e_3^2 + d_1^2, \tag{6}$$
$$|CP|^2 = (1-e_2)^2 + d_2^2. \tag{7}$$

Subtracting (6) from (2), (3) from (4), and (5) from (7) gives

$$|AP|^2 - |CP|^2 = 1 - 2e_3,$$
$$|BP|^2 - |AP|^2 = 1 - 2e_1,$$
$$|CP|^2 - |BP|^2 = 1 - 2e_2.$$

Adding these equations then yields:

$$\sum_{i=1}^{3}(1-2e_i) = 0,$$

or

$$\sum_{i=1}^{3} e_i = \frac{3}{2}. \tag{8}$$

Equally,

$$\sum_{i=1}^{3}(1-e_i) = 3 - \sum_{i=1}^{3} e_i = \frac{3}{2}, \tag{9}$$

and so we have two further invariance properties, (8) and (9). These may be expressed as follows:

$$|AE| + |BF| + |CD| = 3/2,$$
$$|BE| + |CF| + |AD| = 3/2.$$

This is a direct consequence of Carnot's perpendicularity theorem, which states that for *any* triangle ABC, the following holds [1]:

$$|AE|^2 + |BF|^2 + |CD|^2 = |EB|^2 + |CF|^2 + |AD|^2.$$

To see this, simply write this as

$$|AE|^2 - |EB|^2 + |BF|^2 - |CF|^2 + |CD|^2 - |AD|^2 = 0,$$

and use the difference-of-squares formula, while noting that

$$|AE| + |EB| = |BF| + |CF| = |CD| + |AD| = 1.$$

The obvious question is this. Do these invariance properties generalise to regular polygons?

Polygons

We observe immediately that the sum of the distances from an arbitrary interior point P of the unit square is equal to a constant that can be shown to be 2:

$$\sum_{i=1}^{4} d_i = 2. \tag{10}$$

This is readily seen by considering the four differently coloured triangles given by the four corners of the square and the point P and the fact that the area of a triangle is equal to half its height times its base (Figure 4) or, more simply, by observing that $d_1 + d_3 = 1$ and $d_2 + d_4 = 1$.

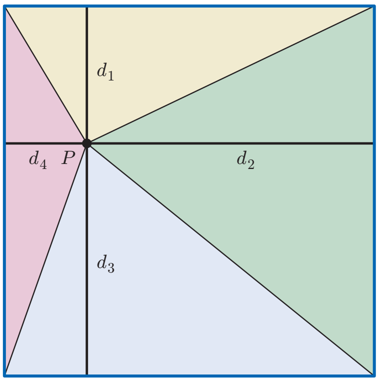

Figure 4: The unit square with lines drawn from an arbitrary point P to each vertex and perpendicular lines drawn to each side.[1]

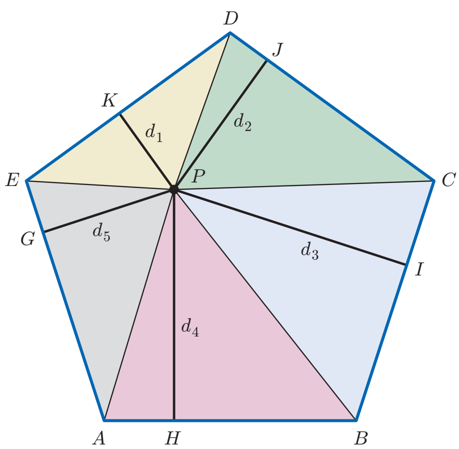

Figure 5: A pentagon (with sides of unit length) with lines drawn from an arbitrary point P to each vertex and perpendicular lines drawn to each side.[1]

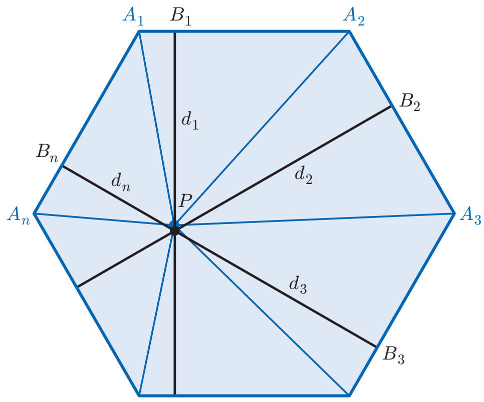

Figure 6: n-sided regular polygon with arbitrary point P.[1]

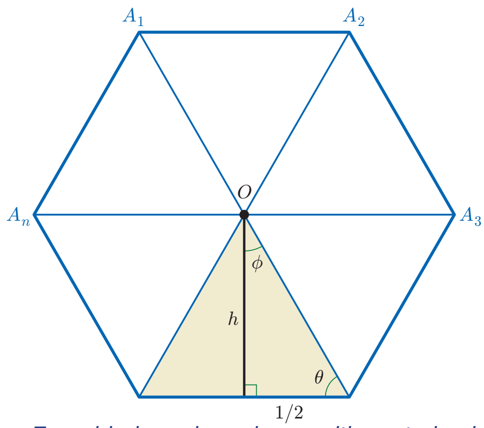

Figure 7: n-sided regular polygon with central point O.[1]

The same is also true for a regular pentagon whose sides are of unit length (Figure 5):

$$\sum_{i=1}^{5} d_i = 2\Delta,$$

where Δ is the area of a regular pentagon.

By the same argument it is clear that the invariance property holds true for any regular polygon with sides of unit length (Figure 6). We need only observe that the area of a regular polygon is equal to the sum of the areas of its triangles:

$$\frac{1}{2}\sum_{i=1}^{n} d_i.$$

To determine the area of an n-sided regular polygon (with sides of unit length), we note that it is simply the combined area of n isosceles triangles, as shown in Figure 7. To determine θ, notice that $\theta + \phi = \pi/2$ (see the lower right triangle in Figure 7). Multiplying by $2n$ gives $2n\theta + 2n\phi = n\pi$. However, $2n\phi = 2\pi$ and so

$$\theta = \left(\frac{n-2}{2n}\right)\pi.$$

Now, since $\tan(\theta) = h/(1/2)$, we have that

$$h = \frac{1}{2}\tan\left(\frac{n-2}{2n}\pi\right).$$

Thus, the area of a regular polygon is

$$\Delta = \frac{n}{2}h = \frac{n}{4}\tan\left(\frac{n-2}{2n}\pi\right).$$

Hence,

$$\sum_{i=1}^{n} d_i = \frac{n}{2}\tan\left(\frac{n-2}{2n}\pi\right), \tag{11}$$

that is,

$$\sum_{i=1}^{n} d_i$$

is a constant. For a regular pentagon:

$$\sum_{i=1}^{n} d_i = \frac{5}{2}\tan\left(\frac{3}{10}\pi\right) = \frac{(5+\sqrt{5})^{3/2}}{4\sqrt{2}}.$$

Note that (11) reduces to (1) and (10) for a triangle and a square, respectively.

Just as for the equilateral triangle (recall Figure 3), this is not the only invariance property associated with a regular polygon: consider Figure 8 and apply Pythagoras' theorem $2n$ times to the triangles A_1PB_1, B_1PA_2, A_2PB_2, etc., to obtain:

$$\begin{aligned}
d_1^2 &= |A_1P|^2 - e_1^2,\\
d_1^2 &= |A_2P|^2 - (1-e_1)^2,\\
d_2^2 &= |A_2P|^2 - e_2^2,\\
d_2^2 &= |A_3P|^2 - (1-e_2)^2,\\
&\cdots\\
d_n^2 &= |A_nP|^2 - e_n^2,\\
d_n^2 &= |A_1P|^2 - (1-e_n)^2.
\end{aligned}$$

Subtracting pairwise and adding the resulting n equations gives

$$\sum_{i=1}^{n}(1-2e_i) = 0,$$

or

$$\sum_{i=1}^{n} e_i = \frac{n}{2}.$$

Furthermore,

$$\sum_{i=1}^{n}(1-e_i) = n - \sum_{i=1}^{n} e_i = n - \frac{n}{2} = \frac{n}{2}.$$

This represents a general invariance property for regular polygons: the equilateral triangle and the square are special cases.

Even for an irregular polygon (Figure 9), we may apply Pythagoras' theorem $2n$ times to the $2n$ right-angled triangles.

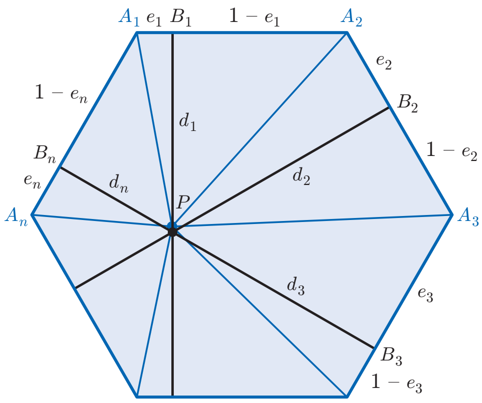

Figure 8: n-sided regular polygon.[1]

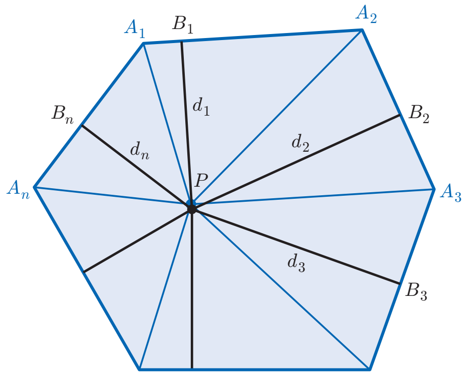

Figure 9: n-sided irregular polygon.[1]

Subtracting pairwise and adding the resulting n equations, we obtain:

$$|A_1B_n|^2 + \sum_{i=1}^{n-1}|A_{i+1}B_i|^2 = \sum_{i=1}^{n}|A_iB_i|^2.$$

This is a natural generalisation of Carnot's perpendicularity theorem (see, e.g. [1]).

Remarks

If, inside the triangle ABC, there is a region for which the sum of the distances from a point P to the sides of the triangle is independent of the position of P, the triangle is equilateral. This statement is the converse of Viviani's theorem and has been considered by Chen and Liang [2]. Three-dimensional generalisations have been studied by de Villiers [3]. A pictorial proof and a proof using vectors have been considered by, respectively, Samuelson [4] and Kawasaki et al. [5]. Further extensions have been put forward by Abboud [6].

An interesting and not totally unrelated problem is that, given a triangle ABC, how can we find a point P for which $|AP| + |BP| + |CP|$ is minimised? This is known as the Fermat–Steiner problem and has been discussed by Gueron and Tessler [7].

Notes

1. Artist: Ron Weickart, Network-Graphics.com

References

1 Wikipedia (2024) *Carnot's theorem (perpendiculars)*, en.wikipedia.org/wiki/Carnot's_theorem_(perpendiculars).

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5 Kawasaki, K.-I., Yagi, Y. and Yanagawa, K. (2005) On Viviani's theorem in 3 dimensions, *Math. Gaz.*, vol. 89, no. 515, p. 283–287.

6 Abboud, E. (2010) Viviani's theorem and its extension, *Coll. Math. J.*, vol. 41, no. 3, pp. 203–211.

7 Gueron, S. and Tessler, R. (2002) The Fermat–Steiner problem, *Am. Math. Month.*, vol. 109, no. 5, pp. 443–451.